Exercises

1. Calculus of variations

Exercise 1.1. (Logan "Applied Mathematics", Exercise 2.13, p. 132.) Consider the functional $J[y] = \int_0^{2\pi} y'^2 dx$. Plot the function $y_0(x) = x$ and the family of curves $y_0(x) + \epsilon h(x)$, where $h(x) = \sin x$. Find $\mathcal{J}(\epsilon) \equiv J[y_0 + \epsilon h]$ and find that $\mathcal{J}'(0) = 0$. This way, you may reach the conclusion that J is stationary with respect to the direction $\sin x$.

Exercise 1.2. (Gelfand, Fomin "Calculus of Variations", Section 4.) Consider the functional

$$J[y] = \int_a^b f(x, y)\sqrt{1 + y'^2}\, dx.$$

(a) Write the Euler-Lagrange equation for this functional. (b) Write the equation as a manifestly second order differential equation. (c) Indicate an example where a functional of this form appears.

Exercise 1.3. (Gelfand, Fomin "Calculus of Variations", Chapter 1, exercise 18.) (a) Find the general solution of Euler's equation corresponding to the functional

$$J[y] = \int_a^b f(x)\sqrt{1 + y'^2}\, dx.$$

(b) Investigate the special cases $f(x) = \sqrt{x}$ and $f(x) = x$.

Exercise 1.4. Assume a Lagrangian

$$L(y) = \frac{1}{2}\dot{y}^2 - U(y)$$

Write the Euler-Lagrange equations for the action $J[y] = \int_a^b L dt$. Find a first integral of the equations.

Extra exercises.

Exercise 1.5. *(Linear functionals)* Logan "Applied Mathematics", Exercise 2.5, p. 131

Exercise 1.6. *(Weak and strong norm)* Logan "Applied Mathematics", Exercise 2.6, p. 131

Exercise 1.7. *(Continuity of linear functionals)* Logan "Applied Mathematics", Exercise 2.9, p. 131

2. Hamiltonian systems

Exercise 2.1. *(Poisson brackets)* Suppose a system with Hamiltonian $H(t, y_i, p_i)$, where t is time. Find a formula for the time derivative of some function $\Phi(y_i, p_i)$ along the solution curves of the system. Write a condition for Φ to be a conserved quantity (i.e., a condition that this time derivative be zero).

Exercise 2.2. *(Harmonic oscillator)* (Gelfand, Fomin "Calculus of Variations", Chapter 4, Exercise 2.) Consider the action functional

$$J[x] = \frac{1}{2}\int_{t_0}^{t_1} (m\dot{x}^2 - kx^2)\, dt.$$

Write the canonical system of the Euler equations corresponding to $J[x]$. Calculate the Poisson brackets $[x, p], [x, H], [p, H]$.

Exercise 2.3. *(Non-Newtonian particles)* Assume two particles on the plane at positions $(x_1, y_1), (x_2, y_2)$ that are functions of time. They interact and the potential of interaction V depends on the distance between the particles $r = \sqrt{(x_2 - x_1)^2 + y_2 - y_1)^2}$. The particles are characterized by charges γ_1, γ_2, and their Lagrangian is given by

$$L = \sum_{i=1}^{2} \frac{\gamma_i}{2}(y_i\dot{x}_i - x_i\dot{y}_i) - V(r).$$

(a) Write the action integral and derive the equations of motion. (b) Write the pairs of canonical position and momenta. (c) Write the Hamiltonian of the system. (d) define the center of charge position for the pair of particles and prove that this is conserved.

Exercise 2.4. *(A nonlinear model for a complex field)* Consider a complex variable Ψ and its dynamical equation given by

$$i\dot{\Psi} = -\omega_0\Psi + g|\Psi|^2\Psi, \qquad \omega_0 : \text{constant}.$$

(a) Show that $|\Psi|$ is constant during the motion. (b) Write a Lagrangian and a Hamiltonian for this system. Also find a pair of conjugate variables (a variable and a conjugate momentum). (c) Find a solution for this nonlinear equation. (d) Add a damping term in the equation.

3. Continuous models

Exercise 3.1. *(Integral for localized solutions)* Consider the Lagrangian denisty

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\partial_x \phi)^2 - U(\phi)$$

where $U(\phi)$ is some potential. (a) Derive the Euler-Lagrange equation. (b) Consider time-independent solutions $\phi = \phi(x)$ and derive a first integral of the equation. [Hint. For the ϕ^4 model we found the integral, for localized solutions, $\frac{1}{2}(\phi')^2 - \lambda(1-\phi^2)^2 = 0$.]

Exercise 3.2. *(Model with quartic potential)* Consider the model as in Exercise ?? with potential

$$U(\phi) = \frac{1}{2}m^2\phi^2 + \frac{1}{2}\lambda\phi^4, \quad m, \lambda > 0.$$

(a) Find one solution for this model. (b) Do we have kink solutions in this model? Give a reason for your answer using the analogy with a mechanical particle moving in a potential.

Exercise 3.3. *(Kink-antikink)* Consider a kink and an antikink solution of the ϕ^4 model, respectively,

$$\phi_k(x) = \tanh\left(\sqrt{2\lambda}(x - x_0)\right), \qquad \phi_a(x) = -\tanh\left(\sqrt{2\lambda}(x - x_0)\right),$$

where x_0 is the center of the kink or antikink. Write a configuration which represents a kink at position a and an antikink at position $-a$. [Hint. This does not need to be a solution of the equation.]

Exercise 3.4. *(Model with three minima)* Consider the model with potential

$$U(\phi) = \frac{1}{2}\phi^2(1-\phi^2)^2.$$

(a) What kind of kinks can we construct in this model? (b) Find such kink solutions.

Extra exercises.

Exercise 3.5. *(Sine-Gordon equation)* (a) Write the sine-Gordon equation and the Lagrangian for this. (b) For the case of time-independent fields, write the sine-Gordon equation in a form of Newton's equation. Give the corresponding potential for the Newton equation. (c) Based on the graph of the potential explain what kind of localized solutions we expect. (d) Derive one of these localised solutions.

Exercise 3.6. *(Bogomolnyi bounds)* (Manton, Satcliffe, "Topological Solitons", Sec. 5.1, 5.2) Assume a theory for a function $\phi(x,t)$ with potential $U(\phi)$. (a) Write the Lagrangian such that a wave equation is produced. (b) Now, focus on static fields $\phi = \phi(x)$ and consider the inequality

$$\left(\frac{1}{\sqrt{2}}\phi' \pm \sqrt{U(\phi)}\right)^2 \geq 0.$$

Take the integral of this in space and derive an inequality for the energy.

Exercise 3.7. *(Euler's equations for the area)* Show that Euler's equation for the functional giving the area of a surface

$$J(u) = \int\int_R \sqrt{1 + u_x^2 + u_y^2}\, dxdy$$

is

$$(1 + u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy} = 0.$$

4. The Gross-Pitaevskii model

Exercise 4.1. *(Focusing nonlinear Schrödinger equation)* Consider $\psi = \psi(x,t)$ satisfying the focusing NLS equation

$$i\dot{\psi} = -\frac{1}{2}\psi'' - \frac{1}{2}|\psi|^2\psi.$$

(a) Verify that the form

$$\psi(x) = \sqrt{2}a\, e^{i\frac{a^2}{2}t}\,\mathrm{sech}(ax)$$

is a time-independent solution, with a an arbitrary constant. Plot this form for two values of a.
(b) Show that, if $\psi(x,t)$ is a solution of the NLS, then a solution traveling with a constant velocity v is

$$\psi_v(x,t) = \exp\left[i\frac{v}{2}\left(x - \frac{v}{2}t\right)\right]\psi(x - vt, t).$$

Write an explicit form of a traveling solution.

Exercise 4.2. *(Defocusing nonlinear Schrödinger equation)* Consider $\psi = \psi(x,t)$ satisfying the defocusing NLS equation

$$i\dot{\psi} = -\frac{1}{2}\psi'' + \frac{1}{2}|\psi|^2\psi.$$

Apply the transformation $\psi(x,t) \to \psi(x,t)e^{-it/2}$ in the above equation.
(a) Verify that the form

$$\psi(x) = \tanh(cx)$$

is a time-independent solution, with c a constant (determine the constant).
(b) Show that a solution traveling with a constant velocity v is of the form

$$\psi(\xi) = ic_1 + c_2\tanh(c\xi), \qquad \xi = x - vt$$

and determine the constants c_1, c_2, c.

Exercise 4.3. *(Gross-Pitaevskii model - rationalizations)* The Gross-Pitaevskii model with a constant potential V, for a field $\Psi = \Psi(\boldsymbol{r}, t)$, is

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Psi - V\Psi + g|\Psi|^2\Psi$$

where $\hbar$ is Planck's constant (with units of energy times time), m is the mass of the particle, g is a nonlinearity parameter and the potential V may be a function of space $V = V(x,y,z)$.
(a) Define new variables for space and time and give a dimensionless form of the model in the case that V is a constant. [Hint. It is enough to do this for one space dimension.]
(b) Assume a harmonic potential

$$V(x,y,z) = \frac{1}{2}m\omega^2(x^2 + y^2)$$

where ω is a frequency. Notice that a new length scale $a = \sqrt{\hbar/(m\omega)}$ is introduced due to the potential (this is called the ”oscillator length”). Use the oscillator length as the unit of length and define new variables to give a dimensionless form for the Gross-Pitaevskii model with the above potential.

Exercise 4.4. *(Continuity equation)* The Gross-Pitaevskii model in a dimensionless form reads

$$i\frac{\partial\Psi}{\partial t} = -\frac{1}{2}\Delta\Psi - \frac{1}{2}(|\Psi|^2 - 1)\Psi.$$

(a) Derive the continuity equation for the Gross-Pitaevskii model. [Hint: The density of mass in the GP model is $|\Psi|^2$.]

(b) Find the expression of the velocity in terms of the argument Θ of the complex field Ψ.
(c) In a two-dimensional model a vortex is of the form (see lecture notes of R. Ricca, Eq. (52))

$$\Psi(r,\theta) = \rho(r)\, e^{\pm i\theta}.$$

Calculate the velocity field for a vortex and show that this form describes flow around the origin.

Exercise 4.5. *(Multivortex)* Consider the polar form of a complex function ψ defined on the plane

$$\psi = \rho\, e^{i\Theta}.$$

and consider the case $\rho = \rho(r)$, $\Theta = \kappa\theta$ where (r,θ) are polar coordinates and $\kappa = \pm 1, \pm 2, \ldots$.
(a) From the Gross-Pitaevskii model for ψ derive an equation for $\rho(r)$.
(b) Solving this equation is not easy. Can you think of some reason about it?
(c) Calculate the velocity field.

Exercise 4.6. *(Gross-Pitaevskii model - dissipative)* Write the Gross-Pitaevskii model with a dissipative term. Prove that this leads to dissipation of energy. [Hint. Write the model in the form $i\frac{\partial\psi}{\partial t} = \frac{\delta E}{\delta\psi^*}$ and work further complementing this form with a dissipative term.]

5. Models in condensed matter

Exercise 5.1. *(Rationalization with an external field)* The energy of a ferromagnet with exchange, easy-axis anisotropy and with an applied field $\boldsymbol{H}$ is

$$E = \frac{A}{M_s^2} \int \partial_\mu \mathbf{M} \cdot \partial_\mu \mathbf{M}\, d^3x + \frac{K}{M_s^2} \int (M_s^2 - M_3^2)\, d^3x - \mu_0 \int \boldsymbol{H} \cdot \mathbf{M}\, d^3x$$

where M_s is the saturation magnetization, A is the exchange parameter, K is the anisotropy parameter and μ_0 is the permeability of vacuum. Define normalized variables and obtain a rationalized form of the energy. [Hint. A field $\boldsymbol{H}$ has the same physical units as $\mathbf{M}$. There are more than one ways to do the normalization and each one gives a different rationalized form. You only need to find one way.]

Exercise 5.2. *(Time-independent Landau-Lifshitz equation)* The rationalized form of the energy of a ferromagnet with exchange and easy-axis anisotropy is (in one space dimension)

$$E = \tfrac{1}{2} \int \partial_x \mathbf{m} \cdot \partial_x \mathbf{m}\, dx + \tfrac{1}{2} \int (1 - m_3^2)\, dx.$$

Derive the time-independent Euler-Lagrange equation given that $\mathbf{m}^2 = 1$.

Exercise 5.3. *(Conserved quantities - discrete)* Assume a spin chain $\mathbf{S}_i$, $i = 1, 2, \cdots, N$ and the equations of motion

$$\dot{\mathbf{S}}_k = \mathbf{S}_k \times \boldsymbol{f}_k, \qquad \boldsymbol{f}_k = -\frac{\partial E}{\partial \mathbf{S}_k}, \quad k = 1, 2, \cdots, N$$

for the energy with exchange and anisotropy,

$$E = -J \sum_{i=1}^{N-1} \mathbf{S}_i \cdot \mathbf{S}_{i+1} + g \sum_{i=1}^{N} [1 - (S_{i,3})^2]$$

where $S_{i,3}$ denotes the third component of $\mathbf{S}_i$. Prove that the system of equations conserve (a) the length of each spin vector $\mathbf{S}_i$, (b) the energy, (c) the total spin $S_3 = \sum_{i=1}^{N} S_{i,3}$. [Hint. You will need to consider carefully the boundary points, especially for question (c).]

Exercise 5.4. *(Conserved quantities - continuum)* Write the Landau-Lifshitz equation

$$\dot{\boldsymbol{m}} = -\boldsymbol{m} \times \boldsymbol{f}, \qquad \boldsymbol{f} = -\frac{\delta E}{\delta \mathbf{m}}$$

for the case of the energy E in Exercise **??** (including exchange and easy-axis anisotropy). Prove that the Landau-Lifshitz equation conserves (a) the length of the local magnetization vector, (b) the energy, (c) the total magnetization $M = \int m_3\, dx$. [Hint. You may give the proofs for a one-dimensional model only.]

Extra.

Exercise 5.5. *(Landau-Lifshitz equation - damping)* The Landau-Lifshitz equation gives the dynamics of the magnetization vector $\boldsymbol{m}$ in a ferromagnet and it is a conservative model

$$\dot{\boldsymbol{m}} = -\boldsymbol{m} \times \boldsymbol{f}$$

where $\boldsymbol{f} = \boldsymbol{f}(\boldsymbol{m})$ is a field whose form depends on the material under study. Write a model for a ferromagnet which is an extension of the above model to include damping effects. That is, the extended model should not be conservative (not conserving energy).

Exercise 5.6. *(Energy of a chiral interaction)* A chiral interaction is one that distinguishes, at each point on a particle chain, the left to the right neighbour. One form of the energy for a chiral interaction in a (one-dimensional) spin chain is the following

$$E_{\mathrm{DM}} = D \sum_i \mathbf{S}_i \times \mathbf{S}_{i+1}.$$

Derive a continuum approximation for E_{DM}. [Hint. Define a continuous field $\mathbf{S}(x)$ and use a Taylor expansion.]

6. Models in Biology

Exercise 6.1. *(SIR model)* Consider the SIR model

$$\dot{S} = -\beta\, SI, \qquad \dot{I} = \beta\, SI - \gamma\, I, \qquad \dot{R} = -\gamma\, I$$

with parameters for infection and recovery β, γ respectively. (a) Derive the curve $I(S)$ and plot it for two values of β (a larger and a smaller one). (b) Determine (analytically or numerically) the effect of β on the peak number of infected, the total number of infected, the time period for the elimination of the disease. (c) In an epidemic, would you propose to isolate parts of the country, e.g., to stop completely communication between the northern and the southern part?

Exercise 6.2. *(Diffusion of insects)* A parasitic worm (Trichostrongylus retortaeformis) grows in areas with sheep and rabbits. The insects spread in a random way in the area and they get eaten by the sheep and rabbits. Consider that the insect population has a constant rate of diffusion D and a death rate proportional to the population (with constant rate per capita μ). [Work for the problem in one spatial dimension only.]
(a) Show that the insect population density $n(x,t)$ obeys a reaction-diffussion equation and write its form. (b) Consider an initial population density $n(x,0) = N_0 \delta(x)$ with $n(\pm\infty, t) = 0$ (no insects are very far away). Verify that the following population density is a solution of the equation

$$n(x,t) = \frac{N_0}{2\sqrt{\pi D t}}\, e^{-\mu t} e^{-x^2/(4Dt)}.$$

(c) Plot the population density distribution for various values of t. Explore various values of the parameters N_0, D, μ. (d) Calculate the integral of $n(x,t)$ over all space. What is the meaning of this integral.

Exercise 6.3. *(Animal dispersal on a plane, Murray Sec. 11.3)* Consider the model for the dispersion of a population density n of animals with diffusion coefficient

$$D(n) = D_0 \left(\frac{n}{n_0}\right)^m$$

where $D_0, n_0 > 0$ constants and $m > 0$ is an integer. The animals live on a plane and they are diffusing only radially. (a) Write the diffusion model. (b) Give the solution of the model. (c) Explain the role of all parameters that are contained in the solution. (d) Give an example by choosing numerical values to the constants and drawing the solution for successive values of time.

Exercise 6.4. *(Fisher-Kolmogoroff equation - rationalization)* The Fisher-Kolmogoroff equation is

$$\frac{\partial n}{\partial t} = rn\left(1 - \frac{n}{n_0}\right) + D\Delta n$$

where $n = n(x,t)$ is the population density and r, n_0, D are positive constants. Define new variables and write the dimensionless form of this equation.

Projects

Project 1. *(Natural boundary conditions)* (Logan "Applied Mathematics" Sec. 3.4 and Gelfand Fomin, Sec. 1.6.)
(a) Study the boundary conditions for equations arising from variational principles.
(b) Study the following example, or find an example of this sort in the literature and study it. A magnetic material is described by the magnetization vector $\mathbf{m} = \mathbf{m}(x)$ with $|\mathbf{m}| = 1$ (in one space dimension). The energy functional for chiral magnets is

$$E = \int \left[\frac{1}{2} (\partial_x \mathbf{m})^2 + \lambda \hat{\mathbf{e}}_y \cdot (\partial_x \mathbf{m} \times \mathbf{m}) \right] dx,$$

where the second term is the chiral term. Use the spherical parametrization for $\mathbf{m}$,

$$m_1 = \sin\Theta \cos\Phi, \quad m_2 = \sin\Theta \sin\Phi, \quad m_3 = \cos\Theta.$$

[(i) Write the energy in terms of (Θ, Φ) - optional]
(ii) Consider the simpler case $\Phi = 0$ and write the energy

$$E = \int \left[\frac{1}{2} (\Theta')^2 - \lambda \Theta' \right] dx$$

(iii) Determine the boundary conditions. (iv) Derive the Hamilton equation. (v) Find a solution that satisfies the equation and the boundary condition.

Project 2. *(Boundary conditions for a magnetic system)* (Logan "Applied Mathematics" Sec. 3.4 and Gelfand Fomin, Sec. 1.6.)
A magnetic material is described by the magnetization vector $\mathbf{m} = (m_1, m_2, m_3)$ where $\mathbf{m} = \mathbf{m}(x)$ with $|\mathbf{m}| = 1$ (in one space dimension). A relevant Hamiltonian in an infinite domain reads

$$E = \int (\partial_x \mathbf{m})^2 \, dx + \lambda \int (m_3 \partial_x m_2 - m_2 \partial_x m_3) \, dx$$

where the second term is the chiral term.
(a) Write the boundary conditions for the problem in the cases (i) $\lambda = 0$ and (ii) $\lambda \neq 0$. (b) Find one nontrivial solution of the Euler equations for $\mathbf{m}$ in the case (ii).

Project 3. *(Bäklund transformation)* (Manton Satcliffe, Sec. 5.3.)
Study Bäcklund transformations. Apply the Bäklund transformation in order to find multi-kink solutions for the sine-Gordon equation.

Project 4. *(Nonlinear Schrödinger equation)*
Derive the Nonlinear Schrödinger equation (NLS) for an atomic, optical or other system. Explain why the NLS models these systems and what the limitations of the model are.

Project 5. *(Bogomolnyi bounds)* (Manton, Satcliffe, "Topological Solitons", Sec. 5.1)
Assume a theory for a function $\phi(x,t)$ with potential $U(\phi)$. (a) Write the Lagrangian such that a wave equation is produced. (b) Focus on static fields $\phi = \phi(x)$ and consider the inequality

$$\left(\frac{1}{\sqrt{2}} \phi' \pm \sqrt{U(\phi)} \right)^2 \geq 0.$$

Take the integral of this in space and derive an inequality for the energy and a corresponding equation (Bogomolny equation) for the miminization of the energy. Use the result in order to derive formal solutions of the model. (c) Choose a specific model and carry out the above explicitely. Here is an example. Consider a vector $\boldsymbol{u} = (u_1, u_2)$ of unit length, $|\boldsymbol{u}| = 1$, on the plane. The energy of the model is

$$E = \int \left[\frac{1}{2} (\partial_x \boldsymbol{u}) \cdot (\partial_x \boldsymbol{u}) + \kappa u_1^2 \right] dx.$$

(i) Find the uniform solutions of this model (states of minimum energy). (ii) Find a kink-type solution using the Bogomolny equation. [Hint. Use the parametrization $u_1 = \cos\phi, u_2 = \sin\phi$.]

Project 6. *(Gross-Pitaevskii model - continuity equation)* (a) Derive the continuity equation for the Gross-Pitaevskii model. [Hint: The density of mass in the GP model is $|\Psi|^2$.]
(b) Find the expression of the velocity in terms of the argument Θ of the complex field Ψ.
(c) In a two-dimensional model a vortex is of the form (see lecture notes of R. Ricca, Eq. (52))

$$\Psi(r, \theta) = \rho(r)\, e^{\pm i\theta}.$$

Calculate the velocity field for a vortex.
(d) A multi-vortex is of the form

$$\Psi(r, \theta) = \rho(r)\, e^{i\kappa\theta}, \qquad \kappa = 1, 2, 3, \ldots.$$

Write the GP model in polar coordinates for such a multi-vortex.
(e) Develop methods or numerical methods in order to find the vortex profile.

Project 7. *(Gross-Pitaevskii model)* Expand upon one of the problems in our common project on the Gross-Pitaevskii model.

Project 8. *(Landau-Lifshitz equation - damping)* The Landau-Lifshitz equation gives the dynamics of the magnetization vector $\boldsymbol{m}$ in a ferromagnet and it is a conservative model

$$\dot{\boldsymbol{m}} = -\boldsymbol{m} \times \boldsymbol{f}$$

where $\boldsymbol{f} = \boldsymbol{f}(\boldsymbol{m})$ is a field whose form depends on the material under study.
(a) White a model for a ferromagnet which is an extension of the above model to include damping effects. That is, the extended model should not be conservative (not conserving energy).
(b) Solve the equation for the case that $\boldsymbol{f}$ is a constant vector (this corresponds to an external field).

Project 9. *(Stereographic projection)* The stereographic projection of a vector $\boldsymbol{m} = (m_1, m_2, m_3)$ which has fixed length $|\boldsymbol{m}| = 1$ is given by the formula

$$u = \frac{m_1 + i m_2}{1 + m_3}.$$

(a) Derive the above formula for the stereographic projection (from the south pole of the sphere). (b) Study the basic theory for the stereographic projection. (c) Consider the Landau-Lifshitz equation and give its basic features (as discussed in the lectures). Find the expression for a magnetic domain wall in terms of the stereographic projection and give its details. (The expression for a domain wall in terms of spherical fields (Θ, Φ) was given in the lecture notes.)

Project 10. *(Equation of motion for the stereographic projection of a field)* The Landau-Lifshitz (LL) equation gives the dynamics of the magnetization vector $\boldsymbol{m}$ in a ferromagnet and it is a conservative model

$$\dot{\boldsymbol{m}} = -\boldsymbol{m} \times \boldsymbol{f}.$$

The magnetization vector has fixed length $|\mathbf{m}| = 1$ and the field in the above equation is $\boldsymbol{f} = \boldsymbol{f}(\boldsymbol{m})$. We may resolve the constraint by using the stereographic projection $u \in \mathbb{C}$ of $\mathbf{m} = (m_1, m_2, m_3)$.
(a) Study the basic theory for the stereographic projection. (b) Derive the formula for u. (c) Write the LL equation in terms of the stereographic projection for the case that $\boldsymbol{f}$ is a constant vector (this corresponds to an external field). (d) Solve the equation for this choice of $\boldsymbol{f}$.

Project 11. *(Fisher–Kolmogorov Equation)* (J.D. Murray, "Mathematical Biology I", Sec. 13.2, 13.3.)
The Fisher–Kolmogorov equation is a reaction-diffusion model that has propagating wave solutions.
(a) Study the model. (b) Find propagating wave solutions. (c) Discuss their application to biological systems.

Project 12. *(Epidemic model with diffusion)* (M. Martcheva, "Introduction to Mathematical Epidemiology', Chapter 15.3.)
Diffusion may be included in an epidemic model (SIR model) to model the spread of the populations.
(a) Derive the equations for such a model as a reaction-diffusion system. (b) Study equilibria of the equations. (c) Find solutions analytically or numerically and discuss their interpretation.

Project 13. *(Epidemic model for a virus)* (M. Martcheva, "Introduction to Mathematical Epidemiology", Sec. 8.)
In an epidemic, consider that some people are infected and develop the disease and some others are infected but do not develop any symptoms.
(a) Write a model for this epidemic, based on the SIR. (b) Make different assumptions and study variations of the model. (c) Find solutions either analytically or numerically. (d) Give an interpretation of the solutions.

Project 14. *(Μελέτη ανάπτυξης όγκων (tumor growth))* ("A Reaction-Diffusion Model of Cancer Invasion", by Gatenby and Gawlinski in Cancer Research, 56, 5745, 1996).)
Στόχος είναι η μαθηματική μοντελοποίηση της ανάπτυξης, εξάπλωσης (μεταστάσεων) καρκινικών όγκων. Αρχικά θα μελετήσετε τα βασικά βιολογικά χαρακτηριστικά εξάπλωσης των καρκινικών όγκων. Κατόπιν θα εξετάσετε και θα παρουσιάσετε ένα κατάλληλο μοντέλο αντίδρασης-διάχυσης.

Project 15. *(Biological Oscillators. FitzHugh-Nagumo Model)* (FitzHugh, 1961; Nagumo, 1962; Murray, 1989, chapter 7) Σε πολλές βιοχημικές αντιδράσεις παρατηρούνται φαινόμενα ταλάντωσης. Ιδιαίτερα σε περιπτώσεις παρουσίας ενζύμων τα οποία παίζουν πολύ σημαντικό ρόλο. (α) Αρχικά θα μελετηθούν γενικά μοντέλα βιολογικών ταλαντωτών. Παράδειγμα: mRNA, Ένζυμο. (β) Κατόπιν θα εξετασθεί πιο αναλυτικά μια περίπτωση, όπως η επικοινωνία μεταξύ νευρώνων μέσω του μοντέλου FitzHugh-Nagumo.

$$\frac{dM}{dt} = \frac{V}{D + P^m} - aM$$
$$\frac{dE}{dt} = bM - cE$$
$$\frac{dP}{dt} = dE - eP.$$