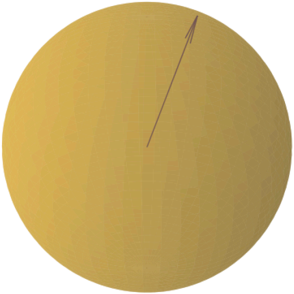
Lecture 6a. A field with values on the sphere, $u(s) \in \mathbb{S}^2$

Assume a field $u = u(s)$ defined in the real space, $s \in \mathbb{R}$, and taking values on the unit sphere, $u \in \mathbb{S}^2$.
Such a field is realised by a vector $\boldsymbol{u} \in \mathbb{R}^3$ with unit length $\boldsymbol{u}^2 = 1$. For any parameter s we have

$$\boldsymbol{u}^2 = 1 \Rightarrow \frac{d}{ds}(\boldsymbol{u} \cdot \boldsymbol{u}) = 0 \Rightarrow \boldsymbol{u} \cdot \frac{d\boldsymbol{u}}{ds} = 0.$$



Any derivative of $\boldsymbol{u}$ is perpendicular to $\boldsymbol{u}$ (it belongs to the tangent plane of the sphere at $\boldsymbol{u}$).

$$\frac{d\boldsymbol{u}}{ds} = \boldsymbol{u} \times \boldsymbol{f}, \quad \text{for some } \boldsymbol{f}.$$

Application. A spin in an external field

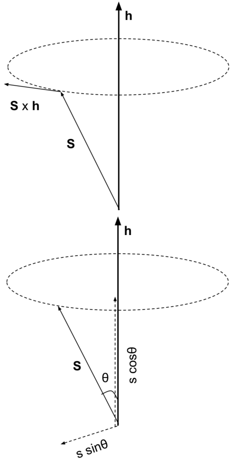
The electrons in the atoms have an internal angular moment, called spin

The spin $\mathbf{S}$ of the atoms (electrons) is a vector of constant length.

The equation of motion for the spin in an external field $\boldsymbol{h}$

$$\frac{d\mathbf{S}}{dt} = \mathbf{S} \times \boldsymbol{h}$$

- $S\cos\theta$: Component of $\mathbf{S}$ parallel to $\boldsymbol{h}$ (remains constant).
- $S\sin\theta$: Projection of $\mathbf{S}$ on the plane perpendicular to $\boldsymbol{h}$ (constant length).
- The moment $\mathbf{S}$ performs precession around $\boldsymbol{h}$.



Energy of a spin

The length of $\mathbf{S}$ is conserved during motion

Note that $|\mathbf{S}|^2 = \mathbf{S} \cdot \mathbf{S}$. We have

$$\frac{d}{dt}(\mathbf{S} \cdot \mathbf{S}) = 2\mathbf{S} \cdot \frac{d\mathbf{S}}{dt} = 2\mathbf{S} \cdot (\mathbf{S} \times \boldsymbol{h}) = 0.$$

Energy of $\mathbf{S}$ in an external magnetic field $\boldsymbol{h}$

$$E = -\mathbf{S} \cdot \boldsymbol{h}.$$

Since $\mathbf{S} = s$ is fixed, the only parameter is the angle θ between $\mathbf{S}$ and $\boldsymbol{h}$,

$$E = -sh\cos\theta.$$

The energy is conserved during motion

$$\frac{dE}{dt} = \frac{dE}{d\mathbf{S}} \cdot \frac{d\mathbf{S}}{dt} = -\boldsymbol{h} \cdot (\mathbf{S} \times \boldsymbol{h}) = 0.$$

Exchange interaction

Ferromagnets

are materials that present non-zero net magnetization (at zero field). This is due to spins of neighbouring electrons that interact and tend to be aligned.

Neighbouring spins are aligned due to exchange interaction.

At the level of two individual spins $\mathbf{S}_1, \mathbf{S}_2$, the energy due to exchange interaction is modelled as

$$-J\,\mathbf{S}_1 \cdot \mathbf{S}_2, \qquad J: \text{exchange constant.}$$

For $J > 0$ and *perfectly aligned spins* the exchange energy has a minimum.

- The exchange interaction induces magnetic order.

Antiferromagnets, Weak Ferromagnets, etc

are materials that present magnetic order (at zero field).

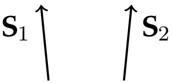
Two spins

In the energy $E = -J\mathbf{S}_1 \cdot \mathbf{S}_2$, each spin plays the role of an external field for the other one. Therefore, the equations of motion are

$$\dot{\mathbf{S}}_1 = J\,\mathbf{S}_1 \times \mathbf{S}_2, \quad \dot{\mathbf{S}}_2 = J\,\mathbf{S}_2 \times \mathbf{S}_1.$$

Note that, the change of the one spin affects the dynamics of the other one. That means that we have a system of coupled equations.

We could loosely imagine that $\boldsymbol{S}_1$ is precessing around $\boldsymbol{S}_2$, while $\boldsymbol{S}_2$ is precessing around $\boldsymbol{S}_1$.



Exercise (Dynamics of two spins)

Study the dynamics of two exchange-coupled spins $\mathbf{S}_1, \mathbf{S}_2$.

A spin chain

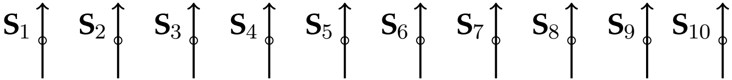
Consider a chain of N spins $\mathbf{S}_i$, $i = 1, 2, \cdots, N$.

The energy of the system is

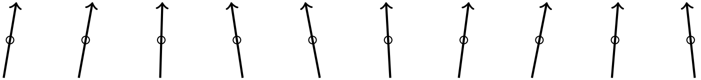
$$E = -J \sum_{i=1}^{N-1} \mathbf{S}_i \cdot \mathbf{S}_{i+1}.$$

Each spin $\mathbf{S}_k$ interacts with two neighbours at $k+1, k-1$.

Chain of aligned spins.



Chain of spins (not fully aligned).



Dynamics of a spin chain

The equation of motion for every $\mathbf{S}_k(t)$ is

$$\dot{\mathbf{S}}_k = J\,\mathbf{S}_k \times (\mathbf{S}_{k+1} + \mathbf{S}_{k-1}), \qquad k = 2, 3, \cdots, N-1.$$

or

$$\dot{\mathbf{S}}_k = \mathbf{S}_k \times \boldsymbol{f}_k, \qquad \boldsymbol{f}_k = -\frac{\partial E}{\partial \mathbf{S}_k}.$$

Exercise (Exchange-coupled spins)

Consider a spin chain and the corresponding system of equations. (a) Specify possible initial conditions and (b) solve the initial value problem numerically.
Consider the following.

- *All spins $\mathbf{S}_i$ should have the same fixed length $|\mathbf{S}_i| = s$.*
- *Try the uniform configuration $\mathbf{S}_i = \mathbf{s}$ for all i, where $\mathbf{s}$ is a constant vector (for example $\mathbf{s} = s\hat{\mathbf{e}}_3$).*
- *Try perturbations of the above uniform configuration.*

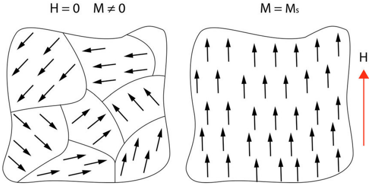
Consider a ferromagnet with aligned magnetic moments.

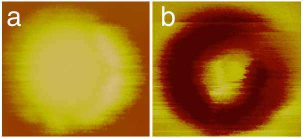
The magnetization is the density of magnetic moments (spin)

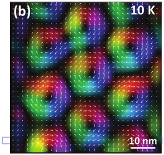
$$\mathbf{M} = \frac{\Delta \boldsymbol{\mu}}{\Delta V}, \qquad \boldsymbol{\mu} \sim \mathbf{S}.$$

By applying a strong magnetic field we may align all magnetic moments ("saturate" the magnetization) along the field direction and measure the saturation magnetization M_s.

Magnetic domains







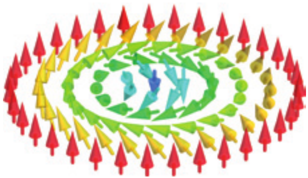
The Bloch sphere

The magnetization vector takes values on the Bloch sphere, $\mathbf{M}^2 = M_s^2$.

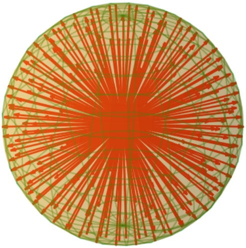
A ferromagnet is described by the magnetization vector $\mathbf{M} = \mathbf{M}(x, t)$ with (see, Landau, Lifshitz, Pitaevskii, "Statistical Physics II")

$$|\mathbf{M}| = M_s \; (=\text{const.}).$$

Magnetization configuration



Bloch sphere



A continuous spin variable

Let us assume a chain of spins which may not be perfectly aligned. The exchange energy depends on the neighbours of each spin $\mathbf{S}_\alpha$,

$$E_{\rm ex} = -J \sum \mathbf{S}_\alpha \cdot \mathbf{S}_{\alpha+1} = -\frac{J}{2} \sum \mathbf{S}_\alpha \cdot (\mathbf{S}_{\alpha+1} + \mathbf{S}_{\alpha-1}).$$

A continuum approximation

Consider a small parameter ϵ and define (ϵ can be defined in different ways)

- A space variable $x = \epsilon\alpha$ where α is an integer index (ϵ may be the spacing between atoms).
- A continuous field $\mathbf{S}(x)$ such that $\mathbf{S}_\alpha = \mathbf{S}(x)$ at the position of each spin α.

The continuous field $\mathbf{S}(x)$ is connecting the discrete spins (atoms) of the material.

Taylor expansion

The advantage of the continuous field is that we can make a

Taylor approximation

When the distance ϵ between spins is small, we have (Taylor expansion)

$$\mathbf{S}_{\alpha\pm1} \approx \mathbf{S} \pm \epsilon\partial_x\mathbf{S} + \frac{\epsilon^2}{2}\partial_x^2\mathbf{S}, \qquad \mathbf{S}_\alpha \to \mathbf{S}.$$

This assumes that

- There is a continuous field $\mathbf{S}(x)$.
- Neighbouring spins differ only a little.

Example (Use the Taylor approximation in the expression for the exchange energy)

$$E_{\mathrm{ex}} = -\frac{J}{2}\sum \mathbf{S}_\alpha \cdot (\mathbf{S}_{\alpha+1} + \mathbf{S}_{\alpha-1}) \approx \cdots$$

Exchange energy (continuum)

Exchange energy

Use the Taylor expansion in the exchange energy

$$E_{\mathrm{ex}} = -J \sum \left(|\mathbf{S}|^2 + \frac{\epsilon^2}{2} \mathbf{S} \cdot \partial_x^2 \mathbf{S} \right) \to -\frac{J}{2} \epsilon \int \mathbf{S} \cdot \partial_x^2 \mathbf{S} \, dx$$

Since $\mathbf{M} \sim \mathbf{S}$ we have $E_{\mathrm{ex}} \sim -\int \mathbf{M} \cdot \partial_x^2 \mathbf{M} \, dx$

and this gives, by a partial integration

$$E_{\mathrm{ex}} = \frac{A}{M_s^2} \int \partial_x \mathbf{M} \cdot \partial_x \mathbf{M} \, dx.$$

- A is the exchange constant (parameter).
- E_{ex} is non-negative.
- Its minimum (perfect alignment, $\partial_x \mathbf{M} = 0$) lies at zero.
- All directions in space, for $\mathbf{M}$, are equivalent.

The nonlinear σ-model

A system with the energy $E_{\rm ex}$ and $\mathbf{M}^2 = $ const. is called the nonlinear σ-model.

Exercise ($O(3)$ invariance of $E_{\rm ex}$)

(α) Write $E_{\rm ex}$ using the components $\mathbf{M} = (M_1, M_2, M_3)$. (b) Consider a uniform rotation for $\mathbf{M}$ and show that $E_{\rm ex}$ remains invariant.

Magnetocrystalline anisotropy

Materials are anisotropic in a natural way, e.g., due to the crystal structure. Anisotropic contributions come from relativistic effects. Some types of anisotropy are simply modelled.

Easy-plane anisotropy

The energy term ($K > 0$ the anisotropy parameter)

$$E_{\mathrm{a}} = \frac{K}{M_s^2} \int (M_3)^2 \, dx$$

favours the states where $\mathbf{M}$ lies on the plane (12), i.e., $M_3 = 0$.

Easy-axis anisotropy

$$E_{\mathrm{a}} = \frac{K}{M_s^2} \int (M_s^2 - M_3^2) \, dx$$

favours the states where $\mathbf{M}$ is fully aligned along the third axis, i.e., $M_3 = \pm M_s$ or $\mathbf{M} = (0, 0, \pm M_s)$.

Examples. Minima of anisotropy energy.

Exercise (Easy-plane anisotropy)

(a) For the easy-plane anisotropy, give all minimum energy solutions.

(b) Show that the energy is invariant with respect to rotations of the vector $\mathbf{M}$ in the (12) plane.

Exercise (Easy-axis anisotropy)

(a) Write the easy-axis anisotropy formula in a manifestly non-negative form to show that $E_{\rm a} \geq 0$.

(b) Give all minimum energy solutions (based on that formula).

Energy and length scales

In three-dimensions (3D), we have the exchange energy

$$E_{\rm ex} = \frac{A}{M_s^2} \int \partial_\mu \mathbf{M} \cdot \partial_\mu \mathbf{M}\, d^3x, \quad \mu = 1, 2, 3.$$

Question (Write explicitly the exchange energy density)

Note that summation is implied for the repeated index μ.

Total energy

In a simple model we assume a ferromagnet with exchange and anisotropy energy. For a 3D magnet,

$$E = E_{\rm ex} + E_{\rm a} = \frac{A}{M_s^2} \int \partial_\mu \mathbf{M} \cdot \partial_\mu \mathbf{M}\, d^3x + \frac{K}{M_s^2} \int (M_s^2 - M_3^2)\, d^3x.$$

Units for the physical constants $A: \mathrm{J/m}$, $K: \mathrm{J/m^3}$.

Dimensional analysis

The length scale of this model

The two energy terms indicate a natural length scale

$$\ell_{\mathrm{DW}} = \sqrt{\frac{A}{K}}.$$

Example (For $A = 10^{-11}\,\mathrm{J/m}, M_s = 10^6\,\mathrm{A/m}, K = 4 \times 10^5\,\mathrm{J/m^3}$)

We calculate $\ell_{\mathrm{DW}} = 5 \times 10^{-9}\,\mathrm{m} = 5\,\mathrm{nm}$.

We define the dimensionless magnetization according to

$$\mathbf{m} = \frac{\mathbf{M}}{M_s}, \qquad \mathbf{m}^2 = 1$$

and we have the energy

$$E = A \int \frac{\partial \mathbf{m}}{\partial x_\mu} \cdot \frac{\partial \mathbf{m}}{\partial x_\mu}\, d^3x + K \int (1 - m_3^2)\, d^3x$$

$$= K \left[\ell_{\mathrm{DW}}^2 \int \frac{\partial \mathbf{m}}{\partial x_\mu} \cdot \frac{\partial \mathbf{m}}{\partial x_\mu}\, d^3x + \int (1 - m_3^2)\, d^3x \right]$$

We define dimensionless space variables (i.e., scale space by $\ell_{\rm DW}$)

$$x_\mu = \xi_\mu \, \ell_{\rm DW}$$

and have the energy

$$E = (K\ell_{\rm DW}^3) \left[\int \partial_\mu \mathbf{m} \cdot \partial_\mu \mathbf{m} \, d^3\xi + \int (1 - m_3^2) \, d^3\xi \right].$$

We write $K\ell_{\rm DW}^3 = A\ell_{\rm DW}$, and re-instate the usual variable $\xi \to x$ to get

$$E = (2A\ell_{\rm DW}) \left[\frac{1}{2} \int \partial_\mu \mathbf{m} \cdot \partial_\mu \mathbf{m} \, d^3x + \frac{1}{2} \int (1 - m_3^2) \, d^3x \right].$$

The natural energy scale is $(2A\ell_{\rm DW})$

Remark

This scaled energy form has no free parameter.

(*) Energy minima

A virial theorem (due to Derrick and Pohozaev) is derived by exploiting the different space scalings of the energy terms.
Assume a D-dimensional space ($D = 1, 2, 3$) and let $\mathbf{m} = \mathbf{m}(\boldsymbol{x})$ correspond to a minimum of the energy. Then, the magnetization configuration $\mathbf{m}(\lambda \boldsymbol{x})$ is scaled (dilated or shrunk) with respect to the energy minimum configuration. For the scaled magnetization we have the energy

$$
\begin{aligned}
E(\lambda) =& \frac{1}{2} \int \partial_\mu \mathbf{m}(\lambda \boldsymbol{x}) \cdot \partial_\mu \mathbf{m}(\lambda \boldsymbol{x}) \, d^D x + \frac{k^2}{2} \int \left[1 - m_3^2(\lambda \boldsymbol{x})\right] \, d^D x \\
=& \lambda^{2-D} \frac{1}{2} \int \partial_\mu \mathbf{m} \cdot \partial_\mu \mathbf{m} \, d^D x + \lambda^{-D} \frac{k^2}{2} \int (1 - m_3^2) \, d^D x
\end{aligned}
$$

Since E is an extremum, it should, in particular, be an extremum with respect to scale transformations, when varying λ, at $\lambda = 1$,

$$
\frac{d}{d\lambda} E(\lambda)\Big|_{\lambda=1} = 0 \Rightarrow (2 - D) E_{\rm ex} = D \, E_{\rm a}.
$$

(*) Virial relation

Result from Derrick scaling argument

$$(2 - D)E_{\rm ex} = D\,E_{\rm a}.$$

Since $E_{\rm ex}, E_{\rm a}$ are positive definite, we conclude

- For $D = 1$, every energy minimum should satisfy $E_{\rm ex} = E_{\rm a}$.
- For $D = 2$, only such energy minima exist for which $E_{\rm a} = 0$, i.e., the solutions are $\mathbf{m} = (0, 0, \pm 1)$.
- For $D = 3$, the equation $-E_{\rm ex} = E_{\rm a}$ does not allow for any energy minimum.

The Derrick-Pohozaev argument only applies to states which give finite energy terms

The field equation is obtained as the Euler-Lagrange equation for the action, with the constraint

$$\mathbf{m}^2 = 1$$

imposed via a Lagrange multiplier. [Raj, Sec. 3.3][FG, Sec. 12.2]
For a demonstration, we consider the exchange interaction and extremize

$$L[\mathbf{m}] = \int d^3x \, \underbrace{\left[\frac{1}{2}\partial_\mu \mathbf{m}\cdot\partial_\mu \mathbf{m} + \frac{\lambda(x)}{2}(1-\mathbf{m}^2)\right]}_{\mathcal{L}}.$$

The Euler-Lagrange equation is

$$-\frac{\delta L}{\delta \mathbf{m}} = 0 \Rightarrow \frac{d}{dx_\mu}\left(\frac{\partial \mathcal{L}}{\partial_\mu \mathbf{m}}\right) - \frac{\partial \mathcal{L}}{\partial \mathbf{m}} = 0.$$

The Landau-Lifshitz equation - pure exchange

We calculate

$$\partial_\mu \partial_\mu \mathbf{m} + \lambda \mathbf{m} = 0 \quad \text{or} \quad \Delta \mathbf{m} + \lambda \mathbf{m} = 0.$$

We multiply the above by $\mathbf{m}$ to obtain the Lagrange multiplier

$$\lambda \mathbf{m} \cdot \mathbf{m} + \mathbf{m} \cdot \Delta \mathbf{m} = 0 \Rightarrow \lambda(x) = -\mathbf{m} \cdot \Delta \mathbf{m}$$

and we use this to eliminate λ in the field equation

$$\Delta \mathbf{m} - (\mathbf{m} \cdot \Delta \mathbf{m})\mathbf{m} = 0 \Rightarrow \mathbf{m} \times (\mathbf{m} \times \Delta \mathbf{m}) = 0.$$

The latter is equivalent to

$$\mathbf{m} \times \Delta \mathbf{m} = 0.$$

Question

Note that $\mathbf{m} \times \Delta \mathbf{m}$ and $\mathbf{m} \times (\mathbf{m} \times \Delta \mathbf{m})$ are both on the plane perpendicular to $\mathbf{m}$. The two vectors are perpendicular to each other.

The Landau-Lifshitz equation - static sector

Form of the Landau-Lifshitz equation

Let us assume an energy functional $E(\mathbf{m})$. Hamilton's equations are

$$\mathbf{m} \times \boldsymbol{f} = 0, \qquad \boldsymbol{f} = -\frac{\delta E}{\delta \mathbf{m}}.$$

- For $\boldsymbol{f} = \boldsymbol{h}$ we recover the standard equation of magnetism for a magnetic moment $\mathbf{m}$ in an external magnetic field $\boldsymbol{h}$.
- Solutions for $\mathbf{m}$ are such that $\mathbf{m} \parallel \boldsymbol{f}(\mathbf{m})$.

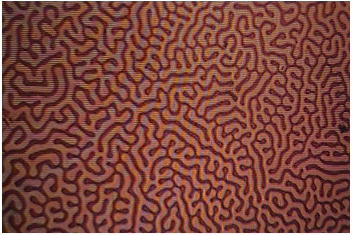
Landau-Lifshitz equation for exchange and easy-axis anisotropy

$$\mathbf{m} \times \left(\Delta \mathbf{m} + k^2 m_3 \hat{\boldsymbol{e}}_3 \right) = 0.$$

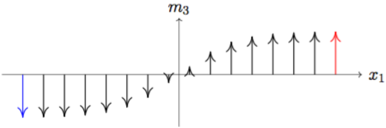
Note: This may also be called a nonlinear σ-model.

One space dimension. A magnetic domain wall.

Domain pattern



Sketch of domain wall



Exercise (Magnetic domain wall formula)

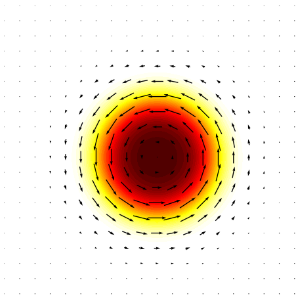
Prove that the configuration

$$m_1 = 0, \qquad m_2 = \frac{1}{\cosh(\lambda x)}, \qquad m_3 = \tanh(\lambda x)$$

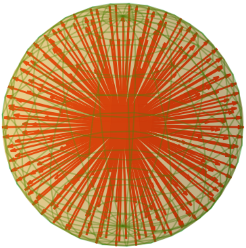
satisfies the 1D Landau-Lifshitz equation with exchange and easy-axis anisotropy for a specific λ.

Two space dimensions (films). A magnetic skyrmion.

A magnetic skyrmion



Bloch sphere



Experimental observation