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On Vector Fields Describing the 2d Motion of a Rigid Body in a Viscous Fluid and Applications

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Abstract. We present some properties of functions in suitable Sobolev spaces which arise naturally in the study of the motion of a rigid body in compressible and incompressible fluid. We relax the regularity assumption of the rigid body by allowing its boundary to be Lipschitz. In the case of a smooth rigid body we obtain a new estimate on the angular velocity. Our results extend and complement related results by V. Starovoitov and moreover we show that they are optimal. As an application we present an example where the rigid body collides with the boundary with non zero speed. Finally, we present a new non collision result concerning a smooth rotating body approaching the boundary, without assuming any special geometry on either the body or the container.

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1. Introduction

Let $\Omega \subset I\!R^2$ be a domain, $S \subset \Omega$ be a bounded connected domain and $\boldsymbol{x}_* \in \overline{S}$ be a fixed point. By $W_0^{1,p}(S, \Omega)$, $p \geq 1$, we denote the vector function space consisting of functions

$$\boldsymbol{u} : \Omega \to I\!R^2,\ \boldsymbol{u} \in \left(W_0^{1,p}(\Omega)\right)^2 \ ,$$

such that for a constant vector $\boldsymbol{a}_* \in I\!R^2$ and a constant $\omega \in I\!R$ there holds

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{a}_* + \omega(\boldsymbol{x} - \boldsymbol{x}_*)^{\perp},\ \ for\ \boldsymbol{x} \in S\ , \tag{1.1}$$

with $\boldsymbol{x} = (x_1, x_2)$, $\boldsymbol{x}^{\perp} = (x_2, -x_1)$ and $\boldsymbol{u} = (u_1, u_2)$.

Such function spaces arise naturally when studying the motion of a rigid body S inside a fluid region Ω, see e.g., [5,8–12]. In this context $\boldsymbol{u}$ is the velocity field in Ω whereas inside the rigid body the velocity is given by (1.1). This is a consequence of the fact that inside the rigid body the deformation tensor is zero, that is,

$$D_{i,j}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = 0,\ \ i,j = 1,2\ . \tag{1.2}$$

The vector fields $\boldsymbol{u}$ for which this holds true, consist of the rigid body vector fields given by (1.1), see e.g., [11,14]. We note that $\boldsymbol{a}_*$ is the linear velocity of the reference point $\boldsymbol{x}_*$ and ω is the angular speed of S around $\boldsymbol{x}_*$.

We are interested both in the static case where the body S touches the boundary $\partial\Omega$ as well as in the dynamic case, where S approaches the boundary $\partial\Omega$. When the boundaries $\partial\Omega$ and ∂S are C^2 such questions have been studied in [8], whereas the case where the boundaries are $C^{1,\alpha}$, $0 < \alpha < 1$, have been studied in [11].

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Suppose that S touches the boundary $\partial\Omega$, that is, $\mathrm{dist}(S, \partial\Omega) = 0$. We further assume that $\boldsymbol{u}$ is solenoidal in Ω that is, $\mathrm{div}\boldsymbol{u}(\boldsymbol{x}) = 0$ for $\boldsymbol{x} \in \Omega$ and moreover $\boldsymbol{u} \in \left(W_0^{1,p}(\Omega)\right)^2$. It is shown in [11], Theorem 2.1, that if both $\partial\Omega$ and ∂S are $C^{1,\alpha}$ with $0 < \alpha < 1$, then

$$p \geq \frac{2+\alpha}{2\alpha} \ \text{ implies } \boldsymbol{a}_* = 0,\ \omega = 0\ .$$

We first show that the above result is optimal, in the sense that for $1 \leq p < \frac{2+\alpha}{2\alpha}$ we construct examples where either ω or $\boldsymbol{a}_*$ are not zero. Next, we relax the smoothness requirement replacing it by Lipschitz continuity of ∂S and $\partial\Omega$. In such a case we show that if $p \geq 2$ then $\boldsymbol{a}_* = 0$, whereas ω does not have to be zero. A similar result is shown in the limit case where S degenerates to a smooth curve, which can be considered as a thin walled solid body, see [4]. This result is optimal and moreover holds true independently of whether $\boldsymbol{u}$ is solenoidal or not in Ω. See Theorem 2.1 for the general case and Theorem 3.2 for the solenoidal case.

We next consider the dynamic case, where S is allowed to move inside Ω. Denote by $S(t) \subset \Omega$ the position of S at time t and set $S_0 = S(0)$. Let us assume that there are $L^\infty(0,T)$ functions $\boldsymbol{a}_*(t)$ and $\omega(t)$ so that $S(t)$ consists of points $\boldsymbol{x}(t)$ satisfying

$$\frac{d\boldsymbol{x}(t)}{dt} = \boldsymbol{a}_*(t) + \omega(t)(\boldsymbol{x}(t) - \boldsymbol{x}_*(t))^\perp,\ \ \boldsymbol{x}(0) = \boldsymbol{x}_0 \in S_0.$$

For $p \geq 1$, $q \geq 1$ we define the space of functions $\boldsymbol{u}(\boldsymbol{x}, t)$, $\boldsymbol{x} \in \Omega$, $t \in (0, T)$,

$$\begin{aligned} &L^q(0,T; W_0^{1,p}(S(t)\,,\Omega)) \\ &:= \{\boldsymbol{u} \in L^q\left(0,T; \left(W_0^{1,p}(\Omega)\right)^2\right):\ \boldsymbol{u}(\boldsymbol{x},t) = \boldsymbol{a}_*(t) + \omega(t)(\boldsymbol{x} - \boldsymbol{x}_*(t))^\perp \ \text{ for } \boldsymbol{x} \in S(t)\}. \end{aligned}$$

We denote by $h(t)$ the distance at time t between $S(t)$ and $\partial\Omega$, that is,

$$h(t) = \mathrm{dist}(S(t), \partial\Omega) = \inf_{\boldsymbol{x} \in S(t)} d(\boldsymbol{x}),$$

where $d(\boldsymbol{x}) = \mathrm{dist}(\boldsymbol{x}, \partial\Omega)$. We assume that S_0 and Ω are $C^{1,\alpha}$ with $0 < \alpha \leq 1$. Then for a solenoidal vector field

$$\boldsymbol{u} \in L^\infty(0,T; L^2(\Omega)) \cap L^1(0,T; W_0^{1,p}(S(t)\,,\Omega)),$$

it is shown in [11], Theorem 3.1, that h is Lipschitz continuous in t and

$$\left|\frac{dh(t)}{dt}\right| \leq C h^{\frac{1+2\alpha}{p(1+\alpha)}\left(p - \frac{2+\alpha}{1+2\alpha}\right)} \|\boldsymbol{u}\|_{W_0^{1,p}(\Omega)} \text{ for } a.a.\ t \in (0,T)\ . \tag{1.3}$$

In the present work we first show that if S is Lipschitz then h is Lipschitz continuous and estimate (1.3) holds with $\alpha = 0$. The same result holds true for a thin walled body as well. In addition, we do not require $\boldsymbol{u}$ to be solenoidal, see Theorem 4.3 for the precise statement.

Next, under the assumption that S is $C^{1,\alpha}$ with $0 < \alpha \leq 1$, we obtain the following estimate for the angular velocity ω:

$$|\omega| \leq C h^{\frac{2\alpha}{p(1+\alpha)}\left(p - \frac{2+\alpha}{2\alpha}\right)} \|\nabla \boldsymbol{u}\|_{L^p(\Omega)}; \tag{1.4}$$

see Theorem 4.6 for the precise statement. This estimate is new and complements (1.3). Moreover, we show that this estimate as well as (1.3) are optimal.

In the context of rigid body motion in a viscous incompressible fluid, Starovoitov [11] shows that if $\boldsymbol{u}$ is in the usual energy space and $\alpha \geq \frac{1}{2}$ then the body comes to the boundary of Ω with zero speed. We construct an example which shows that for $0 \leq \alpha < \frac{1}{2}$ collision is possible with a non zero speed, see Example 1 of Sect. 5.

Hillairet [7] has shown that if the rigid body S is a disc and Ω is half–plane, then in the absence of external forces there is no collision. In [6] Gérard–Varet and Hillairet show that for a $C^{1,\alpha}$ body moving vertically near a flat horizontal part of $\partial\Omega$, under the action of gravity, collision is possible if and only if

$\alpha < \frac{1}{2}$. It is important to note that in [6,7] the authors use essentially the Navier–Stokes equations and not just the fact that $\boldsymbol{u}$ is a member of suitable Sobolev spaces.

Using (1.4) we present a formal argument suggesting that for smooth S and Ω and in the absence of external forces, collision is impossible. The argument is formal since we assume that the fluid is governed by the stationary Stokes equations for a.a. t. On the other hand no special geometry on S or Ω is assumed.

The paper is organized as follows. In Sect. 2 we consider the case where S and Ω are in contact, that is, $\mathrm{dist}(S, \partial\Omega) = 0$ without requiring the vector field $\boldsymbol{u}$ to be solenoidal. The solenoidal case is considered in Sect. 3. In Sect. 4 we consider the dynamic case where S is approaching the boundary of Ω. In Sect. 5 several examples are presented and discussed. Finally in Sect. 6 the non collision result is presented.

2. The General Case During Contact

In this section we assume that the rigid body $S \subset \Omega$ touches the boundary of Ω at the point P. Without loss of generality we take $P = (0,0) = \boldsymbol{x}_*$. In particular we have

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{a} + \omega \boldsymbol{x}^\perp, \quad \text{for } \boldsymbol{x} \in S \ ,$$

where for simplicity of notation we write $\boldsymbol{a}$ instead of $\boldsymbol{a}_*$. By "general case" we mean that $\boldsymbol{u}$ is not necessarily divergence free.

We consider two cases. First we consider the case where $\partial\Omega$ and ∂S are both $C^{1,\alpha}$ with $0 < \alpha < 1$. In this case we choose our coordinate system x_1x_2 so that the x_1 axis coincides with the common tangent to ∂S and $\partial\Omega$ at the point P. One easily checks that there exist $k > 0$ and $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$,

$$\begin{aligned} P^+_{\varepsilon,\alpha} &:= \{|x_1| < \varepsilon,\ k|x_1|^{1+\alpha} < x_2 < k\varepsilon^{1+\alpha}\} \subset S \ , \\ P^-_{\varepsilon,\alpha} &:= \{|x_1| < \varepsilon,\ -k\varepsilon^{1+\alpha} < x_2 < -k|x_1|^{1+\alpha}\} \subset I\!R^2 \backslash \overline{\Omega} \ . \end{aligned} \tag{2.1}$$

We also denote by $\Pi_{\varepsilon,\alpha}$ the orthogonal parallelogram

$$\Pi_{\varepsilon,\alpha} := \{(x_1, x_2):\ |x_1| < \varepsilon,\ |x_2| < k\varepsilon^{1+\alpha}\} \ . \tag{2.2}$$

In the second case $\partial\Omega$ and ∂S are Lipschitz. We recall that if Ω is a Lipschitz domain and $\boldsymbol{x} \in \partial\Omega$, it follows, see [2], Chapter 4, that there exists a finite sector $Sec_0(0, r_0, \theta)$ contained in $\Omega^c = I\!R^2 \backslash \Omega$. Here $Sec_0(0, r_0, \theta) = B(0, r_0) \cap Sec(0, \theta_0)$ where $B(0, r_0)$ is the open ball centered at $(0,0)$ of radius r_0 and $Sec(0, \theta_0)$ is the angle with vertex at $(0,0)$ and opening θ_0.

We then have

Theorem 2.1. *Let $\boldsymbol{u} \in W^{1,p}_0(S, \Omega)$.*

(A) Suppose first that Ω, S are $C^{1,\alpha}$, with $\alpha \in (0,1)$.

(i) If $p \geq \frac{2+\alpha}{1+\alpha}$ then $\boldsymbol{a} = 0$, moreover if $p \geq \frac{2+\alpha}{\alpha}$ then $\omega = 0$ as well.

This result is optimal in the following sense

(ii) There exist Ω, S and $\boldsymbol{u} \in W^{1,p}_0(S, \Omega)$ such that if $1 \leq p < \frac{2+\alpha}{1+\alpha}$ then $\boldsymbol{a} \neq 0$, whereas if $1 \leq p < \frac{2+\alpha}{\alpha}$ then $\omega \neq 0$.

(B) Suppose now that Ω, S are Lipschitz continuous.

(iii) If $p \geq 2$ then $\boldsymbol{a} = 0$.

This is optimal in the following sense

(iv) there exist domains Ω, S and a function $\boldsymbol{u} \in W^{1,p}_0(S, \Omega)$ such that if $1 \leq p < 2$ then $\boldsymbol{a} \neq 0$, whereas if $p \geq 1$ then $\omega \neq 0$.

Proof. Throughout the proof we extent $\boldsymbol{u}$ outside Ω by zero.

A (i) Triangle inequality gives that

$$\left(\int_{P^+_{\varepsilon,\alpha}} |\boldsymbol{u}|^p d\boldsymbol{x}\right)^{\frac{1}{p}} = \left(\int_{P^+_{\varepsilon,\alpha}} |\boldsymbol{a}+\omega\boldsymbol{x}^\perp|^p d\boldsymbol{x}\right)^{\frac{1}{p}}$$
$$\geq |\boldsymbol{a}|\left(\int_{P^+_{\varepsilon,\alpha}} 1 d\boldsymbol{x}\right)^{\frac{1}{p}} - |\omega|\left(\int_{P^+_{\varepsilon,\alpha}} |\boldsymbol{x}|^p d\boldsymbol{x}\right)^{\frac{1}{p}}$$
$$\geq \left| c_1|\boldsymbol{a}|\varepsilon^{\frac{2+\alpha}{p}} - c_2|\omega|\varepsilon^{1+\frac{2+\alpha}{p}}\right| , \tag{2.3}$$

for some positive constants c_1, c_2 independent of ε. On the other hand Poincare's inequality in $\Pi_{\varepsilon,\alpha}$ gives that for some positive constant c_0, independent of ε,

$$\left(\int_{\Pi_{\varepsilon,\alpha}} |\boldsymbol{u}|^p d\boldsymbol{x}\right)^{\frac{1}{p}} \leq c_0\varepsilon^{1+\alpha}\left(\int_{\Pi_{\varepsilon,\alpha}} |\nabla\boldsymbol{u}|^p d\boldsymbol{x}\right)^{\frac{1}{p}} , \tag{2.4}$$

where $\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})^T$, $\boldsymbol{u} = (u_1, u_2)$ and $\nabla\boldsymbol{u}$ is the corresponding tensor. From (2.3) and (2.4) we get that

$$\left| c_1|\boldsymbol{a}|\varepsilon^{\frac{2+\alpha}{p}} - c_2|\omega|\varepsilon^{1+\frac{2+\alpha}{p}}\right| \leq c_0\varepsilon^{1+\alpha}\left(\int_{\Pi_{\varepsilon,\alpha}} |\nabla\boldsymbol{u}|^p d\boldsymbol{x}\right)^{\frac{1}{p}} ,$$

from which it follows

$$\left| c_1|\boldsymbol{a}| - c_2|\omega|\varepsilon\right| \leq c_0\varepsilon^{1+\alpha-\frac{2+\alpha}{p}}\left(\int_{\Pi_{\varepsilon,\alpha}} |\nabla\boldsymbol{u}|^p d\boldsymbol{x}\right)^{\frac{1}{p}} , \tag{2.5}$$

Sending ε to zero we get that $\boldsymbol{a} = 0$ provided that $p \geq \frac{2+\alpha}{1+\alpha}$. Setting $\boldsymbol{a} = 0$ in (2.5), simplifying and sending ε to zero we conclude that $\omega = 0$ when $p \geq \frac{2+\alpha}{\alpha}$.

(ii) Let Ω be the half disc

$$\Omega = \{(x_1, x_2) : \ x_1^2 + x_2^2 < R^2, \ x_2 > 0\} ,$$

of which we smooth out the boundary and S be the domain bounded by the curves $x_2 = k|x_1|^{1+\alpha}$, $k > 0$, and $x_1^2 + x_2^2 = \rho^2$, $x_2 > 0$, $2\rho < R$, for which domain we also smooth out the boundary.

Since both domains are symmetric with respect to the x_2–axis we restrict ourselves to the first quadrant, that is, the $x_1 > 0$, $x_2 > 0$ part of the plane. We next define the function

$$\boldsymbol{u}(\boldsymbol{x}) = \psi(|\boldsymbol{x}|)\phi\left(\frac{x_2}{kx_1^{1+\alpha}}\right)\boldsymbol{u}_S(\boldsymbol{x}) , \tag{2.6}$$

where,

$$\boldsymbol{u}_S(\boldsymbol{x}) = \boldsymbol{a} + \omega\boldsymbol{x}^\perp, \ \boldsymbol{x} \in I\!\!R^2 \ , \ \boldsymbol{a} \neq 0,$$

and $\psi(r)$ and $\phi(\tau)$ are smooth cutoff functions given by

$$\psi(r) = \begin{cases} 1, & 0 < r < \rho, \\ C^\infty, & \rho < r < 2\rho \ , \\ 0, & 2\rho < r \ , \end{cases} \quad \phi(\tau) = \begin{cases} 1, & \tau \geq 1, \\ C^\infty, & 0 < \tau < 1 \\ 0, & \tau \leq 0. \end{cases} \tag{2.7}$$

It is easily seen that $\boldsymbol{u}$ is a bounded C^1 function in the upper half plane. Moreover,

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{a} + \omega\boldsymbol{x}^\perp \text{ for } \boldsymbol{x} \in S \text{ and } \boldsymbol{u}(\boldsymbol{x}) = 0 \text{ on } \partial\Omega.$$

In the sequel we will check the integrability of $|\nabla\boldsymbol{u}|^p$. Taking the gradient of $\boldsymbol{u}$ we have the following matrix equality

$$\nabla\boldsymbol{u} = \nabla\psi\phi\boldsymbol{u}_S + \psi\nabla\phi\boldsymbol{u}_S + \psi\phi\nabla\boldsymbol{u}_S \ . \tag{2.8}$$

The first and third terms of the right hand side are bounded, hence the integrability of the left hand side is reduced to the integrability of the middle term of the right hand side. Concerning the middle term we easily note that

(a) its integrability is decided by the term $\nabla\phi = \nabla\phi\left(\frac{x_2}{kx_1^{1+\alpha}}\right)$, the other two factors being bounded,

(b) it is enough to check the domain of integration

$$0 < x_1 < \sigma,\ 0 < x_2 < kx_1^{1+\alpha}\ ,$$

for σ small .

We easily see

$$|\nabla\phi| \leq Cx_1^{-1-\alpha},\ 0 < x_1 < \sigma,\ 0 < x_2 < kx_1^{1+\alpha}\ ,$$

and

$$\int_0^\sigma \int_0^{kx_1^{1+\alpha}} |\nabla\phi|^p\ dx_2 dx_1 \leq C\int_0^\sigma x_1^{(1+\alpha)(p-1)} dx_1. \tag{2.9}$$

Then for $\boldsymbol{a} \neq 0$, we have $\boldsymbol{u} \in W_0^{1,p}(\Omega;\mathbb{R}^2)$ provided that $p < \frac{2+\alpha}{1+\alpha}$.

Suppose now that $\frac{2+\alpha}{1+\alpha} \leq p < \frac{2+\alpha}{\alpha}$. We then have $\boldsymbol{a} = 0$ and therefore

$$|\boldsymbol{u}_S(\boldsymbol{x})| = |\omega||\boldsymbol{x}|,\ \omega \neq 0\ .$$

The integrability of $|\nabla\boldsymbol{u}|^p$ is now equivalent to the finiteness of

$$\int_0^\sigma \int_0^{kx_1^{1+\alpha}} |x|^p x_1^{-(1+\alpha)p} dx_2 dx_1,$$

which is true provided that $p < \frac{2+\alpha}{\alpha}$.

B (iii) Since Ω is Lipschitz continuous there exists a finite sector $Sec_0(0,\theta,\frac{\varepsilon}{\sin\theta}) \subset \Omega^c$, $0 < \varepsilon < \varepsilon_0$. We choose our orthogonal system to be x_1x_2 where x_2 is along the direction of the bissectrice of the angle θ. We also define

$$\Pi_\varepsilon := \{(x_1,x_2):\ |x_1| < \varepsilon,\ |x_2| < \varepsilon\cot\theta\}\ . \tag{2.10}$$

Since S is Lipschitz there exists a finite sector $Sec_0(0,\tilde{r}_0,\tilde{\theta}_0) \subset S$. We also set $S_\varepsilon = Sec_0(0,\tilde{r}_0,\tilde{\theta}_0) \cap \Pi_\varepsilon$. We work as in A(i)

$$\begin{aligned}\left(\int_{S_\varepsilon} |\boldsymbol{u}|^p d\boldsymbol{x}\right)^{\frac{1}{p}} &\geq |\boldsymbol{a}|\left(\int_{S_\varepsilon} 1 d\boldsymbol{x}\right)^{\frac{1}{p}} - |\omega|\left(\int_{S_\varepsilon} |\boldsymbol{x}|^p d\boldsymbol{x}\right)^{\frac{1}{p}} \\ &\geq \left|c_1|\boldsymbol{a}|\varepsilon^{\frac{2}{p}} - c_2|\omega|\varepsilon^{1+\frac{2}{p}}\right|\ ,\end{aligned}$$

where we used that $|S_\varepsilon| = c_0\varepsilon^2$ and $|\boldsymbol{x}| < c_1\varepsilon$ for suitable positive constants c_0, c_1. We conclude as in the proof of A(i) with $\alpha = 0$ there.

(iv) Here we use the construction of part (ii) with $\alpha = 0$. We omit further details. □

What happens if the rigid body degenerates to a line segment L touching the boundary $\partial\Omega$? This can be thought as a thin wall body approximation, see e.g. [4]. In this case we understand the requirement $\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{a} + \omega\boldsymbol{x}^\perp$ for $x \in S$ in the sense of trace. As in the previous case we asume that L touches the boundary of Ω at the point $(0,0)$.

We then have

Theorem 2.2. *Let* $\boldsymbol{u} \in \left(W_0^{1,p}(\Omega)\right)^2$ *and* $\mathrm{Tr}\big|_L \boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{a} + \omega\boldsymbol{x}^\perp$.

A. Assume that $\partial\Omega$ *is Lipschitz continuous. If* $p \geq 2$ *then* $\boldsymbol{a} = 0$.

B. Assume that $\partial\Omega$ *is* $C^{1,\alpha}$, $\alpha \in (0,1]$, *and that* L *is* **tangent** *to* $\partial\Omega$. *If* $p \geq \frac{2+\alpha}{1+\alpha}$ *then* $\boldsymbol{a} = 0$, *moreover if* $p \geq \frac{2+\alpha}{\alpha}$ *then* $\omega = 0$ *as well.*

C. All the above results are optimal similarly to Theorem 2.1.

In case (A) above, ω is not necessarily zero, no matter what p is. Similarly, in case (B) if L touches $\partial\Omega$ nontangentially then by part (A), $\boldsymbol{a} = 0$ provided that $p \geq 2$ and ω is not necessarily zero no matter what p is.

Proof. The proof is similar to the proof of Theorem 2.1. We assume that the point of contact is $(0, 0)$.

A. Let Π_ε be the orthogonal defined in (2.10) centered at $(0, 0)$ and let $L_\varepsilon = L \cap \Pi_\varepsilon$. Similarly to Theorem 2.1 the result is optimal.

Triangle inequality and simple estimates give that

$$\left(\int_{L_\varepsilon} |\boldsymbol{u}|^p ds\right)^{\frac{1}{p}} \geq c(|\boldsymbol{a}|\varepsilon^{\frac{1}{p}} - |\omega|\varepsilon^{1+\frac{1}{p}}) \ . \tag{2.11}$$

where we used $|L_\varepsilon| \sim \varepsilon$.

Assume at the moment that $\varepsilon = 1$. By the standard trace theorem we have that

$$\left(\int_{L_1} |\boldsymbol{u}|^p ds\right)^{\frac{1}{p}} \leq c_1 \left(\int_{\Pi_1} |\boldsymbol{u}|^p d\boldsymbol{x}\right)^{\frac{1}{p}} + c_1 \left(\int_{\Pi_1} |\nabla \boldsymbol{u}|^p d\boldsymbol{x}\right)^{\frac{1}{p}} \ .$$

On the other hand by Poincare inequality

$$\left(\int_{\Pi_1} |\boldsymbol{u}|^p d\boldsymbol{x}\right)^{\frac{1}{p}} \leq c_2 \left(\int_{\Pi_1} |\nabla \boldsymbol{u}|^p d\boldsymbol{x}\right)^{\frac{1}{p}} \ .$$

Combining the previous two estimates we get that

$$\left(\int_{L_1} |\boldsymbol{u}|^p ds\right)^{\frac{1}{p}} \leq c_3 \left(\int_{\Pi_1} |\nabla \boldsymbol{u}|^p d\boldsymbol{x}\right)^{\frac{1}{p}} \ .$$

Scaling the above inequality to Π_ε we finally have

$$\left(\int_{L_\varepsilon} |\boldsymbol{u}|^p ds\right)^{\frac{1}{p}} \leq c_3 \varepsilon^{\frac{p-1}{p}} \left(\int_{\Pi_\varepsilon} |\nabla \boldsymbol{u}|^p d\boldsymbol{x}\right)^{\frac{1}{p}} \ . \tag{2.12}$$

From (2.11) and (2.12) we conclude

$$c|\boldsymbol{a}| - c|\omega|\varepsilon \leq c_3 \varepsilon^{\frac{p-2}{p}} \left(\int_{\Pi_\varepsilon} |\nabla \boldsymbol{u}|^p d\boldsymbol{x}\right)^{\frac{1}{p}} \ ,$$

and the result follows taking $\varepsilon \to 0$.

B. We now have that L is part of the x_1 axis and let $\Pi_{\varepsilon,\alpha}$ be as defined in (2.2). We also set $L_\varepsilon = L \cap \Pi_{\varepsilon,\alpha}$ and $\Pi^-_{\varepsilon,\alpha} = \Pi_{\varepsilon,\alpha} \cap \{x_2 < 0\}$. For $x_1 \in L_\varepsilon$ we easily have

$$\Big| |\boldsymbol{a}| - |\omega||x_1| \Big| \leq |\boldsymbol{u}(x_1, 0)| \leq \int_{-k|x_1|^{1+\alpha}}^{0} |\boldsymbol{u}_{x_2}(x_1, x_2)| dx_2.$$

Integrating with respect to $x_1 \in L_\varepsilon$ we obtain

$$\Big| c_1|\boldsymbol{a}|\varepsilon - c_2|\omega|\varepsilon^2 \Big| \leq \int_{\Pi^-_{\varepsilon,\alpha}} |\nabla \boldsymbol{u}| d\boldsymbol{x} \leq \varepsilon^{(2+\alpha)\frac{p-1}{p}} \|\nabla \boldsymbol{u}\|_{L^p(\Pi^-_{\varepsilon,\alpha})},$$

for suitable positive constants c_1, c_2 independent of ε. We conclude as before.

C. For the optimality we use a similar construction as in Theorem 2.1(ii). ☐

Remark 2.1. Clearly the same proof works if instead of a line segment we have a smooth finite curve.

3. The Solenoidal Case During Contact

In this section we continue to assume that the rigid body $S \subset \Omega$ touches the boundary of Ω at the point $P = (0,0) = \boldsymbol{x}_*$, but we impose the extra assumption that $\boldsymbol{u}$ is divergence free i.e., $\mathrm{div}\boldsymbol{u} = 0$. We continue to use the same coordinate system as in Sect. 2. If we denote by $\boldsymbol{n}_P$ and $\boldsymbol{\tau}_P$ the outward (to S) unit normal and unit tangential vector respectively and by $\boldsymbol{u}_P$ the velocity of the rigid body at the point P we have

$$\boldsymbol{u}_P = \boldsymbol{a} = (a_1,\ a_2) = (\boldsymbol{u}_P \cdot \boldsymbol{\tau}_P,\ -\boldsymbol{u}_P \cdot \boldsymbol{n}_P). \tag{3.1}$$

We first present an auxiliary Lemma for later use

Lemma 3.1. *Let $\boldsymbol{u} \in W_0^{1,p}(S,\Omega)$, $p > 1$, with $\mathrm{div}\boldsymbol{u} = 0$. Suppose that Ω, S are $C^{1,\alpha}$, with $\alpha \in (0,1)$. Then there exist positive constants C and ε_0 such that for all $\varepsilon \in (0,\varepsilon_0)$ there holds*

$$|a_2 - \frac{2k}{2+\alpha} a_1 \varepsilon^\alpha - \omega \left(\frac{1}{3} + \frac{k^2 \varepsilon^{2\alpha}}{3+2\alpha} \right) \varepsilon| \le C(\|\nabla \boldsymbol{u}\|_{L^p(\Pi_{\varepsilon,\alpha})} \varepsilon^\alpha + \|\mathrm{div}\boldsymbol{u}\|_{L^p(\Pi_{\varepsilon,\alpha})}) \varepsilon^{1+\alpha-\frac{\alpha+2}{p}}\ . \tag{3.2}$$

Proof. We denote by $\Gamma_\rho^\pm$ the graph of the curves $x_2 = \pm k x_1^{1+\alpha}$, $0 \le x_1 \le \rho \le \varepsilon_0$ and by $\mathcal{G}_\rho$ the planar region enclosed by the curves Γ_ρ^+, Γ_ρ^- and the line $x_1 = \rho$. We note that on Γ_ρ^+ we have $\boldsymbol{u} = \boldsymbol{a} + \omega \boldsymbol{x}^\perp$ whereas $\boldsymbol{u} = 0$ on Γ_ρ^-. By the divergence theorem applied to $\mathcal{G}_\rho$ we have

$$\int_{\mathcal{G}_\rho} \mathrm{div}\boldsymbol{u}\ dx_1 dx_2 = \int_{-k\rho^{1+\alpha}}^{k\rho^{1+\alpha}} u_1(\rho, x_2)\, dx_2 + \int_{\Gamma_\rho^+} \boldsymbol{u} \cdot \boldsymbol{n}\, ds\ ,$$

where $\boldsymbol{n}$ is the outward unit normal to $\mathcal{G}_\rho$ and the line integrals are taken counter clockwise. Straightforward calculations show that

$$a_2 \rho - a_1 k \rho^{1+\alpha} - \frac{1}{2}\omega(\rho^2 + k^2 \rho^{2+2\alpha}) = \int_{-k\rho^{1+\alpha}}^{k\rho^{1+\alpha}} u_1(\rho, x_2)\, dx_2 - \int_{\mathcal{G}_\rho} \mathrm{div}\boldsymbol{u}\ dx_1 dx_2\ .$$

We now integrate the last equality from $\rho = 0$ to $\rho = \varepsilon \le \varepsilon_0$ to get

$$\begin{aligned} &\frac{1}{2} a_2 \varepsilon^2 - \frac{k}{2+\alpha} a_1 \varepsilon^{2+\alpha} - \omega \left(\frac{1}{6} + \frac{k^2 \varepsilon^{2\alpha}}{6+4\alpha} \right) \varepsilon^3 \\ &\quad = \int_{\mathcal{G}_\varepsilon} u_1(x_1, x_2)\, dx_1 dx_2 - \int_0^\varepsilon \int_{\mathcal{G}_\rho} \mathrm{div}\boldsymbol{u}\ dx_1 dx_2 d\rho\ . \end{aligned} \tag{3.3}$$

We next estimate the two integrals in the right hand side. Using Holder and Poincare inequalities as well as the fact that $|\mathcal{G}_\varepsilon| = \frac{2k}{2+\alpha}\varepsilon^{2+\alpha}$ we obtain

$$\begin{aligned} |\int_{\mathcal{G}_\varepsilon} u_1(x_1, x_2)\, dx_1 dx_2| &\le \|\boldsymbol{u}\|_{L^p(\Pi_{\varepsilon,\alpha})} |\mathcal{G}_\varepsilon|^{\frac{p-1}{p}} \\ &\le C \|\nabla \boldsymbol{u}\|_{L^p(\Pi_{\varepsilon,\alpha})}\, \varepsilon^{3+2\alpha-\frac{\alpha+2}{p}}\ . \end{aligned} \tag{3.4}$$

On the other hand

$$\begin{aligned} |\int_0^\varepsilon \int_{\mathcal{G}_\rho} \mathrm{div}\boldsymbol{u}\ dx_1 dx_2 d\rho| &\le \int_0^\varepsilon \left(\int_{\mathcal{G}_\rho} |\mathrm{div}\boldsymbol{u}|^p\ dx_1 dx_2 \right)^{\frac{1}{p}} |\mathcal{G}_\rho|^{\frac{p-1}{p}} d\rho \\ &\le \varepsilon \left(\int_{\mathcal{G}_\varepsilon} |\mathrm{div}\boldsymbol{u}|^p\ dx_1 dx_2 \right)^{\frac{1}{p}} |\mathcal{G}_\varepsilon|^{\frac{p-1}{p}} \\ &\le C \|\mathrm{div}\boldsymbol{u}\|_{L^p(\Pi_{\varepsilon,\alpha})}\, \varepsilon^{3+\alpha-\frac{\alpha+2}{p}}\ . \end{aligned} \tag{3.5}$$

From (3.3)–(3.5) the result follows. □

We then have

Theorem 3.2. *Let $\boldsymbol{u} \in W_0^{1,p}(S,\Omega)$ be such that* $\mathrm{div}\boldsymbol{u} = 0$.

(A) Suppose first that Ω, S are $C^{1,\alpha}$, with $\alpha \in (0,1)$.

(i) If $p \geq \frac{2+\alpha}{1+2\alpha}$ then $a_2 = 0$.

(ii) If $p \geq \frac{2+\alpha}{1+\alpha}$ then $\boldsymbol{a} = 0$.

(iii) If $p \geq \frac{2+\alpha}{2\alpha}$ then $\boldsymbol{a} = 0$ and $\omega = 0$.

The above results are optimal in the following sense:

(iv) there exist Ω, S and $\boldsymbol{u} \in W_0^{1,p}(S,\Omega)$ such that
if $1 \leq p < \frac{2+\alpha}{2\alpha}$ then $\boldsymbol{a} \neq 0$,
if $\frac{2+\alpha}{2\alpha} \leq p < \frac{2+\alpha}{1+\alpha}$ then $a_1 \neq 0$,
if $\frac{2+\alpha}{1+\alpha} \leq p < \frac{2+\alpha}{2\alpha}$ then $\omega \neq 0$.

(B) Suppose now that Ω, S Lipschitz continuous.

(v) If $p \geq 2$ then $\boldsymbol{a} = 0$. This is optimal in the sense that there exist domains Ω, S and a function $\boldsymbol{u} \in W_0^{1,p}(S,\Omega)$ such that if $1 \leq p < 2$ then $\boldsymbol{a} \neq 0$, whereas if $p \geq 1$ then $\omega \neq 0$.

Proof. (i) Using Lemma (3.1) with $\mathrm{div}\boldsymbol{u} = 0$ and sending ε to zero we get that $a_2 = 0$.

(ii) This is a consequence of Theorem 2.1(ii). Alternatively from (3.2), after setting $\mathrm{div}\boldsymbol{u} = 0$ and $a_2 = 0$ and simplifying we get

$$| - \frac{2k}{2+\alpha} a_1 \ + \omega \left(\frac{1}{3} + \frac{k^2 \varepsilon^{2\alpha}}{3+2\alpha} \right) \varepsilon^{1-\alpha} | \leq C \| \nabla \boldsymbol{u} \|_{L^p(\Pi_{\varepsilon,\alpha})} \, \varepsilon^{1+\alpha-\frac{\alpha+2}{p}} \ . \tag{3.6}$$

Sending ε to zero we get that $a_1 = 0$.

(iii) This is Theorem 2.1 in [11]. Alternatively, it follows from (3.6) by a similar procedure as in (ii).

(iv) Let Ω and S be as in the proof of Theorem 2.1(ii). We recall

$$\boldsymbol{u}_S(\boldsymbol{x}) = \boldsymbol{a} + \omega \boldsymbol{x}^\perp, \ \boldsymbol{x} \in I\!R^2 \ ,$$

and define

$$\Phi(\boldsymbol{x}) := \boldsymbol{a} \cdot \boldsymbol{x}^\perp + \frac{1}{2} \omega |\boldsymbol{x}|^2 \ ,$$

so that

$$\boldsymbol{u}_S(\boldsymbol{x}) = (u_{1S}, \, u_{2S}) = \nabla^\perp \Phi(\boldsymbol{x}) = \left(\frac{\partial \Phi}{\partial x_2}, -\frac{\partial \Phi}{\partial x_1} \right) \ .$$

With ψ and ϕ as in (2.7) we take

$$\boldsymbol{u}(\boldsymbol{x}) = \nabla^\perp \left(\psi(|\boldsymbol{x}|) \phi \left(\frac{x_2}{k x_1^{1+\alpha}} \right) \Phi(\boldsymbol{x}) \right) \ . \tag{3.7}$$

Clearly $\mathrm{div}\boldsymbol{u} = 0$. Since $\psi = 1$ for $|\boldsymbol{x}| < \rho$, we have that for such $\boldsymbol{x}$

$$\nabla \boldsymbol{u} = \Phi \nabla \nabla^\perp \phi + \nabla \Phi \nabla^\perp \phi + \nabla \phi \nabla^\perp \Phi + \phi \nabla \nabla^\perp \Phi \ .$$

We easily see that in the region $0 \leq x_1 \leq \rho$, $0 \leq x_2 \leq k x_1^{1+\alpha}$ the following estimates hold true for $i, j = 1, 2$

$$|\phi| < C, \ |\phi_{x_i}| < C x_1^{-1-\alpha}, \ |\phi_{x_i x_j}| < C x_1^{-2-2\alpha}$$

$$|\Phi| < C x_1 (|\boldsymbol{a}| + |\omega| x_1) \ |\Phi_{x_i}| < C(|\boldsymbol{a}| + |\omega| x_1), \ |\Phi_{x_i x_j}| < C|\omega|.$$

We then compute

$$|\nabla \boldsymbol{u}| \leq C(|\boldsymbol{a}| + |\omega| x_1) x_1^{-1-2\alpha} \ .$$

As a consequence the integrability of $|\nabla \boldsymbol{u}|^p$ in Ω is equivalent to the finiteness of the integral

$$\int_0^\rho \int_0^{k x_1^{1+\alpha}} (|\boldsymbol{a}| + |\omega| x_1)^p x_1^{-p(1+2\alpha)} \, dx_2 dx_1 \ .$$

If $|\boldsymbol{a}| \neq 0$ the above integral is finite for

$$p < \frac{2+\alpha}{1+2\alpha} \, .$$

If on the other hand $|\boldsymbol{a}| = 0$ and $\omega \neq 0$ then the integral is finite for

$$p < \frac{2+\alpha}{2\alpha} \, .$$

Now choosing $\Phi(\boldsymbol{x}) = a_1 x_2$, with $a_1 \neq 0$, we note that

$$|\Phi_{x_i}| < C x_1^{1+\alpha}, \ i = 1, 2, \ 0 < x_2 < k x_1^{1+\alpha},$$

and therefore $|\nabla \boldsymbol{u}| \leq C x_1^{-1-\alpha}$. This time the integrability of $|\nabla \boldsymbol{u}|^p$ is ensured by the condition

$$p < \frac{2+\alpha}{1+\alpha} \, .$$

(v) The fact that if $p \geq 2$ then $\boldsymbol{a} = 0$ is shown in Theorem 2.1 (B) without the requirement $\mathrm{div}\boldsymbol{u} = 0$. For the optimality we use Ω and S as in Theorem 2.1 and $\boldsymbol{u}$ as given in (3.7) with $\alpha = 0$. As in part (iv) we obtain

$$|\nabla \boldsymbol{u}| \leq C(|\boldsymbol{a}| + |\omega| x_1) x_1^{-1} \, .$$

We conclude as before. □

Remark 3.1. We recall that if both S and Ω are differentiable at the point $P = (0, 0)$ then the x_1 axis is along the common tangent to ∂S and $\partial\Omega$. If we denote by $\boldsymbol{n}_P$ and $\boldsymbol{\tau}_P$ the outward to S unit normal and unit tangential vector respectively and by $\boldsymbol{u}_P$ the velocity of the rigid body at the point P we have

$$(a_1, \ a_2) = \boldsymbol{a} = \boldsymbol{u}_P = (\boldsymbol{u}_P \cdot \boldsymbol{\tau}_P, \ -\boldsymbol{u}_P \cdot \boldsymbol{n}_P).$$

We next consider the case of the thin wall approximation. Assume that the line segment L touches the boundary $\partial\Omega$.

Theorem 3.3. *Let* $\boldsymbol{u} \in \left(W_0^{1,p}(\Omega)\right)^2$ *be such that* $\mathrm{div}\boldsymbol{u} = 0$ *and* $\mathrm{Tr}\big|_L \boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{a} + \omega \boldsymbol{x}^\perp$.

A. Assume that $\partial\Omega$ *is Lipschitz continuous. If* $p \geq 2$ *then* $\boldsymbol{a} = 0$.

B. Assume that $\partial\Omega$ *is* $C^{1,\alpha}$, $\alpha \in (0, 1]$, *and that* L *is* ***tangent*** *to* $\partial\Omega$. *If* $p \geq \frac{2+\alpha}{1+2\alpha}$ *then* $a_2 = 0$. *If* $p \geq \frac{2+\alpha}{1+\alpha}$ *then* $\boldsymbol{a} = 0$. *Finally if* $p \geq \frac{2+\alpha}{2\alpha}$ *then* $\omega = 0$ *as well.*

C. All the above results are optimal similarly to Theorem 3.2.

In case (A) above, ω is not necessarily zero, no matter what p is. Similarly, in case (B) if L touches $\partial\Omega$ nontangentially then by part (A), $\boldsymbol{a} = 0$ provided that $p \geq 2$ and ω is not necessarily zero no matter what p is.

Proof. Part A is true (without u being divergent free) by Theorem 2.2. For part B we take L to lie on the x_1 axis and let $\Pi^-_{\varepsilon,\alpha}$ be as defined in (2.2). Let $\mathcal{T}_\rho$ be the region enclosed by the x_1 axis, the parabola Γ^-_ρ and the vertical line $x_1 = \rho$. Noting that on L we have $\boldsymbol{u} = (a_1, a_2 - \omega x_1)$ and applying divergence Theorem in $\mathcal{T}_\rho$ we get

$$a_2 \rho - \frac{1}{2}\omega \rho^2 = \int_{-k\rho^{1+\alpha}}^{0} u_1(\rho, x_2) dx_2.$$

Integrating the above from $\rho = 0$ to ε and using standard inequalities as before we arrive at

$$\left| \frac{1}{2}|a_2|\varepsilon^2 - \frac{1}{6}|\omega|\varepsilon^3 \right| \leq \varepsilon^{3+2\alpha-\frac{2+\alpha}{p}} \|\nabla \boldsymbol{u}\|_{L^p(\mathcal{T}_\varepsilon)}, \tag{3.8}$$

from which the result follows in case $p \geq \frac{2+\alpha}{1+2\alpha}$. The case where $p \geq \frac{2+\alpha}{1+\alpha}$ is a consequence of Theorem 2.2. Finally from (3.8) after setting $a_2 = 0$ there follows that $\omega = 0$ when $p \geq \frac{2+\alpha}{2\alpha}$.

For the optimality we use a similar construction as in Theorem 3.2. □

4. Before the Contact

In this Section we consider the case where S is allowed to move. By $S = S(t)$ we denote the position of the rigid body at time $t \in [0, T)$, with $S(0) = S_0$ and by $h(t)$ the distance at time t between $S(t)$ and $\partial\Omega$. Throughout the Section we assume that h is strictly positive.

We say that a domain $D \subset I\!R^2$ is *uniformly* $C^{1,\alpha}$, $\alpha \in (0, 1]$, if there exist constants $k > 0$ and $\rho_0 > 0$ s.t. for each $x_0 \in D$ there is an isometry $\mathcal{R} : I\!R^2 \to I\!R^2$ s.t.

$$\mathcal{R}(0) = x_0,\ \ \mathcal{R}(\{x \in I\!R^2 :\ k|x_1|^{1+\alpha} < x_2 < k|\rho_0|^{1+\alpha},\ |x_1| < \rho_0\}) \subset D.$$

We note that if D is a bounded $C^{1,\alpha}$ domain then it is a uniformly $C^{1,\alpha}$ domain, see e.g, [1].

We recall the following result by Starovoitov

Theorem 4.1 [11]. *Let $\Omega \subset I\!R^2$ and $S_0 \subset \Omega$ be uniformly $C^{1,\alpha}$, $\alpha \in (0, 1]$ domains, S_0 being in addition bounded and*

$$\boldsymbol{u} \in L^\infty(0, T; (L^2(\Omega))^2) \cap L^1(0, T; W_0^{1,p}(S(t), \Omega)),\ \ p \in [1, \infty],\ \ \mathrm{div}\boldsymbol{u} = 0.$$

We assume that $h(t) < H$ for some $H > 0$. Then $h(t)$ is Lipschitz continuous and there exists a constant C depending only on S, Ω, H and p such that

$$\left|\frac{dh(t)}{dt}\right| \le Ch^\beta \|\nabla \boldsymbol{u}\|_{L^p(\Omega)} \text{ for } a.e.\ t \in (0, T); \tag{4.1}$$

here

$$\beta = \frac{1 + 2\alpha}{p(1 + \alpha)} \left(p - \frac{2 + \alpha}{1 + 2\alpha}\right).$$

The following is a direct consequence of the above result, see Theorem 3.2 in [11]

Corollary 4.2. *Let the conditions of Theorem 4.1 be satisfied and in addition*

$$\boldsymbol{u} \in L^q(0, T; W_0^{1,p}(S(t), \Omega)),\ \ p, q \in [1, \infty].$$

If $h(t_*) = 0$ for some $t_* \in [0, T]$ and $\beta < 1$, then there exists a positive function $\varepsilon(t)$ such that

$$h(t) = \varepsilon^2(t)(t_* - t)^{\frac{q-1}{q(1-\beta)}},\ \ t \in (0, t_*),\ \ \varepsilon(t) \xrightarrow[t\uparrow t_*]{} 0. \tag{4.2}$$

In this section we first extend Theorem 4.1 for a Lipschitz rigid body. We next complement estimate (4.1) with a similar estimate for the angular velocity $\omega(t)$.

4.1. Lipschitz Rigid Body

Here we consider the case of a Lipschitz rigid body as well as the limiting case where the rigid body degenerates to a smooth curve. The main result is

Theorem 4.3. *Let $\Omega \subset I\!R^2$ be a domain that satisfies a uniform two sided ball condition of radius R and $S_0 \subset \Omega$ be either a bounded Lipschitz domain or a finite smooth curve. Let*

$$\boldsymbol{u} \in L^\infty(0, T; (L^2(\Omega))^2) \cap L^1(0, T; W_0^{1,p}(S(t), \Omega)),\ \ p \in [2, \infty].$$

Then $h(t)$ is Lipschitz continuous and there exists a constant C depending only on S, Ω, H and p such that

$$\left|\frac{dh(t)}{dt}\right| \le Ch^{\frac{p-2}{p}} \|\boldsymbol{u}\|_{W_0^{1,p}(\Omega)} \text{ for } a.e.\ t \in (0, T)\ . \tag{4.3}$$

Remark 4.1. In case Ω is bounded the uniform two sided ball condition is equivalent to Ω being a $C^{1,1}$ domain see, e.g., [1], Corollary 3.14.

Remark 4.2. We note that in Theorem 4.1, if we set $\alpha = 0$ then $\beta = \frac{p-2}{p}$. The proof of [11] however does not work for $\alpha = 0$. In addition, we do not require $\boldsymbol{u}$ to be solenoidal; see also Example 4 of next Section.

To prove the above Theorem we will combine two auxiliary lemmas. We first recall a few facts. In case $S \subset \Omega$ is a bounded Lipschitz domain, it follows, see [2], Chapter 4, that S satisfies the *uniform* cone condition, that is, there exists a length r_0 and an angle $\theta_0 \in (0, \frac{\pi}{3})$ such that for any point $\boldsymbol{x} \in \partial S$ a finite sector $Sec_0(\boldsymbol{x}, r_0, \theta)$ is contained in $\bar{S}$. The sector $Sec_0(\boldsymbol{x}, r_0, \theta)$ is given, modulo translation and rotation, by $B(\boldsymbol{x}, r_0) \cap Sec(\boldsymbol{x}, \theta_0)$ where $B(\boldsymbol{x}, r_0)$ is the open ball centered at $\boldsymbol{x}$ of radius r_0 and $Sec(\boldsymbol{x}, \theta_0)$ is the angle with vertex at $\boldsymbol{x}$ and opening θ_0.

Let $P \in \partial S$ and $Q \in \partial\Omega$ be two points that realize the distance h between S and $\partial\Omega$, that is

$$h = |PQ| = \text{dist}(S, \partial\Omega) > 0 \ .$$

We choose our coordinate system $x_1 x_2$ s.t. Q is the origin $(0, 0)$ and P is the point $(0, h) = \boldsymbol{x}_P$. We also note that the x_1–axis is tangent to $\partial\Omega$ at the point Q. Our first Lemma refers to a fixed time t and $S = S(t)$. For simplicity we drop the t dependence and write $\boldsymbol{u}(\boldsymbol{x})$ instead of $\boldsymbol{u}(\boldsymbol{x}, t)$. We then have

Lemma 4.4. *Let $\Omega \subset I\!R^2$ be a domain that satisfies a uniform two sided ball condition of radius R and $S \subset \Omega$ be either a bounded Lipschitz domain or a finite smooth curve. Let $\boldsymbol{u} \in W_0^{1,p}(S, \Omega)$, $p > 1$ and $\boldsymbol{u}_P$ be the value of the vector field $\boldsymbol{u}$ at the point P. Then, for $h < R$ we have*

$$|\boldsymbol{u}_P| \leq C h^{\frac{p-2}{p}} \left(\|\nabla \boldsymbol{u}\|_{L^p(\Omega)} + h^{\frac{2}{p}} \|\boldsymbol{u}\|_{L^2(S)} \right) \ .$$

where $C = C(S, R)$ is a constant that depends only on S and R.

Proof. **A**. Suppose first that S is a bounded Lipschitz domain. By our assumptions we have that when $h < R$ and k is such that $0 < k < \min\{1, \frac{r_0}{R}\}$ then $Sec_0(P, kh, \theta) \subset \bar{S}$. Define

$$\Pi_h = \{(x_1, x_2) : \ -h < x_1 < h, \ -h < x_2 < 2h\}.$$

Since $P = (0, h)$ and $k < 1$ there holds $Sec_0(P, kh, \theta) \subset \Pi_h$. Using the fact that in the rigid body $\boldsymbol{u} = \boldsymbol{u}_P + \omega(\boldsymbol{x} - \boldsymbol{x}_P)^\perp$ and triangle inequality we have that

$$\begin{aligned} |\boldsymbol{u}_P| \left(\frac{\theta_0 (kh)^2}{2} \right)^{\frac{1}{p}} - |\omega| \left(\frac{\theta_0 (kh)^{p+2}}{p+2} \right)^{\frac{1}{p}} &\leq \|\boldsymbol{u}\|_{L^p(Sec_0(P, kh, \theta))} \\ &\leq \|\boldsymbol{u}\|_{L^p(\Pi_h)} \\ &\leq c_0 h \|\nabla \boldsymbol{u}\|_{L^p(\Pi_h)}; \end{aligned}$$

the last inequality follows from Poincare inequality since the bottom side of Π_h is outside of Ω and u is zero there. It then follows

$$|\boldsymbol{u}_P| \leq C h^{\frac{p-2}{p}} (\|\nabla \boldsymbol{u}\|_{L^p(\Pi_h)} + h^{\frac{2}{p}} |\omega|). \tag{4.4}$$

From the fact that $\boldsymbol{u} \in L^\infty(0, T; (L^2(\Omega))^2)$ and (1.1) there follows that

$$|\omega| \leq C_0 \|\boldsymbol{u}\|_{L^2(S)}, \quad \text{with } C_0 = C_0(S);$$

see, e.g., [11] p. 321, and the result follows.

B. Suppose that S is a finite smooth curve which for simplicity we take to be a line segment L. With P, Q and Π_h as before we set $L_h = \Pi_h \cap L$. On L we have $\boldsymbol{u} = \boldsymbol{u}_P + \omega(\boldsymbol{x} - \boldsymbol{x}_P)^\perp$ and by triangle inequality and simple estimates we get

$$\left| c_1 |\boldsymbol{u}_P| h^{\frac{1}{p}} - c_2 |\omega| h^{1+\frac{1}{p}} \right| \leq \left(\int_{L_h} |\boldsymbol{u}_P|^p ds \right)^{\frac{1}{p}} ,$$

For suitable positive constants independent of h. On the other hand similarly to estimate (2.12),

$$\left(\int_{L_h} |\boldsymbol{u}|^p ds \right)^{\frac{1}{p}} \leq c_3 h^{\frac{p-1}{p}} \left(\int_{\Pi_h} |\nabla \boldsymbol{u}|^p d\boldsymbol{x} \right)^{\frac{1}{p}} \ .$$

From these two estimates we obtain (4.4) and conclude as before. □

We next state our second Lemma. We now allow time to evolve. We will use the following notation: By $\boldsymbol{x}(t, P(t_0))$ we denote the trajectory that at time $t_0 \in [0,T]$ passes through the point $P(t_0) \in \overline{S}(t_0)$, that is the solution of the ODE

$$\frac{d\boldsymbol{x}(t)}{dt} = \boldsymbol{a}_*(t) + \omega(t)(\boldsymbol{x}(t) - \boldsymbol{x}_*(t))^{\perp}, \tag{4.5}$$

where $\boldsymbol{x}(t_0)$ is the radius vector of the point $P(t_0) \in \overline{S}(t_0)$. With this notation we have that

$$h(t) = \inf_{P(t_0)\in\overline{S}(t_0)} d(\boldsymbol{x}(t, P(t_0))) \ , \ \forall \ t, \ t_0 \in [0,T]. \tag{4.6}$$

The following result does not require any smoothness from the part of S.

Lemma 4.5. *Assume that Ω satisfies the inner sphere condition of radius $R > 0$. Suppose that at time t_0 a point $P_m(t_0) \in \overline{S}(t_0)$ with radius vector $\boldsymbol{x}(t_0, P_m(t_0))$ realizes the distance of $S(t_0)$ to $\partial\Omega$, that is*

$$h(t_0) = d(\boldsymbol{x}(t_0, P_m(t_0))).$$

Let $h < R$. Then $h(t)$ is a Lipschitz continuous function and for almost all t_0 we have

$$\frac{dh(t)}{dt}\Big|_{t=t_0} = \nabla_{\boldsymbol{x}} d(\boldsymbol{x}(t_0, P_m(t_0))) \cdot \frac{d\boldsymbol{x}}{dt}(t, P_m(t_0)))\Big|_{t=t_0} . \tag{4.7}$$

In particular, if ∂S is differentiable at the point $P = P_m(t_0)$ and $\boldsymbol{n}_P$ is the outward unit normal there, then

$$\frac{dh(t)}{dt}\Big|_{t=t_0} = -\boldsymbol{n}_P \cdot \boldsymbol{u}_P. \tag{4.8}$$

Proof. We first recall that $d(\boldsymbol{x})$ is 1-Lipschitz function in Ω and a C^1 function in $\Omega_R := \{\boldsymbol{x} \in \Omega : \ d(\boldsymbol{x}) < R\}$. Because of (4.5) and the fact that $\boldsymbol{a}_*$ and ω are bounded (see [11]) it follows that the function $\boldsymbol{x}(t, P(t_0))$ is a Lipschitz continuous function of the first variable and as a consequence the composed function $d(\boldsymbol{x}(t, P(t_0)))$ is Lipschitz continuous in $t \in (0,T)$. It then follows that $h(t)$ as given by (4.6) is Lipschitz continuous in $t \in (0,T)$ as well and therefore differentiable for a.e. $t \in (0,T)$. On the other hand, *all trajectories* defined by (4.5) are easily seen to be differentiable at the Lebesgue points of $\boldsymbol{a}_*(t)$ and $\omega(t)$, and the same is true for the functions $d(\boldsymbol{x}(t, P(t_0)))$ and in particular for $d(\boldsymbol{x}(t, P_m(t_0)))$. Since almost all points of $\boldsymbol{a}_*(t)$ and $\omega(t)$ in $(0,T)$ are Lebesgue points we conclude that at the exception of a set of measure zero both functions $h(t)$ and $d(\boldsymbol{x}(t, P_m(t_0)))$ are differentiable. Let t_0 be a point where both $h(t)$ and $d(\boldsymbol{x}(t, P_m(t_0)))$ are differentiable. For $\tau > 0$ we have

$$\begin{aligned} h(t_0+\tau) - h(t_0) &= \inf_{P\in\overline{S}(t_0)} d(\boldsymbol{x}(t_0+\tau, P(t_0))) - \inf_{P\in\overline{S}(t_0)} d(\boldsymbol{x}(t_0, P(t_0))) \\ &= \inf_{P\in\overline{S}(t_0)} d(\boldsymbol{x}(t_0+\tau, P(t_0))) - d(\boldsymbol{x}(t_0, P_m(t_0))) \\ &\le d(\boldsymbol{x}(t_0+\tau, P_m(t_0))) - d(\boldsymbol{x}(t_0, P_m(t_0))) \ , \end{aligned}$$

from which it follows that

$$\frac{dh(t)}{dt}\Big|_{t=t_0} \le \frac{d}{dt} d(\boldsymbol{x}(t, P_m(t_0)))\Big|_{t=t_0} .$$

Similarly we have that

$$h(t_0) - h(t_0-\tau) \ge d(\boldsymbol{x}(t_0, P_m(t_0))) - d(\boldsymbol{x}(t_0-\tau, P_m(t_0))) \ ,$$

from which it follows that

$$\frac{dh(t)}{dt}\Big|_{t=t_0} \ge \frac{d}{dt} d(\boldsymbol{x}(t, P_m(t_0)))\Big|_{t=t_0} ,$$

and therefore

$$\begin{aligned} \frac{dh(t)}{dt}\Big|_{t=t_0} &= \frac{d}{dt} d(\boldsymbol{x}(t, P_m(t_0)))\Big|_{t=t_0} \\ &= \nabla_{\boldsymbol{x}} d(\boldsymbol{x}(t_0, P_m(t_0))) \cdot \frac{d\boldsymbol{x}}{dt}(t, P_m(t_0))\Big|_{t=t_0} , \end{aligned}$$

and this completes the proof of (4.7). Relation (4.8) follows from (4.7), taking into account that

$$\nabla_{\boldsymbol{x}} d(\boldsymbol{x}(t_0, P_m(t_0)) = \frac{\overrightarrow{QP}}{|QP|} = (0,\, 1) = -\boldsymbol{n}_P$$

see e.g., Theorem 2.2.7 in [3] and

$$\frac{d\boldsymbol{x}}{dt}(t, P_m(t_0))\Big|_{t=t_0} = \boldsymbol{u}_{P_m(t_0)}. \tag{4.9}$$

□

We are now ready to give the proof of Theorem 4.3.

Proof of Theorem 4.3. : Since $\boldsymbol{u} \in L^1(0,T; W_0^{1,p}(S(t),\Omega))$ we have that for almost all $t_0 \in (0,T)$ function $\boldsymbol{u}(\cdot, t_0)$ is an element of $W_0^{1,p}(S(t_0),\Omega)$. At such a time t_0, we have (4.9) and

$$\nabla_{\boldsymbol{x}} d(\boldsymbol{x}(t_0, P_m(t_0)) = \frac{\overrightarrow{QP}}{|QP|} = (0,\, 1).$$

Now the result follows from Lemmas 4.4 and 4.5 taking also into account that

$$\|\boldsymbol{u}\|_{L^2(S)} \le |S|^{\frac{p-2}{2p}} \|\boldsymbol{u}\|_{L^p(S)} \le |S|^{\frac{p-2}{2p}} \|\boldsymbol{u}\|_{L^p(\Omega)}.$$

□

Remark 4.3. If in the above proof we use (4.4) in the place of the estimate of Lemma 4.4 we have that for $p \ge 2$

$$\left|\frac{dh(t)}{dt}\right| \le Ch^{\frac{p-2}{p}} \left(\|\nabla \boldsymbol{u}\|_{L^p(\Pi_h)} + h^{\frac{2}{p}}|\omega|\right), \text{ for } a.e.\ t \in (0,T); \tag{4.10}$$

here the orthogonal parallelogram Π_h is defined in the proof of Lemma 4.4. In particular $\dot{h} \to 0$ when $h \to 0$, even for $p = 2$. Moreover, when $\omega = 0$, $|\dot{h}|$ is bounded by a local norm of $\nabla\boldsymbol{u}$, in the sense that in the right hand side of (4.10) appears the integral of $\nabla\boldsymbol{u}$ over Π_h instead of Ω.

In the smooth case where ∂S, $\partial\Omega$ are $C^{1,\alpha}$ with $\alpha > 0$, examination of the proof of [11] Theorem 3.1, for $n = 2$, shows that in fact the following local estimate is true

$$\left|\frac{dh(t)}{dt}\right| \le Ch^{\beta} \|\nabla \boldsymbol{u}\|_{L^p(\mathcal{G}_{h,\sigma_0})} \text{ for } a.e.\ t \in (0,T); \tag{4.11}$$

here $\sigma_0 = ch^{\frac{1}{1+\alpha}}$ for a suitable positive constant c and

$$\mathcal{G}_{h,\sigma} = \{(x_1, x_2) : -kx_1^{1+\alpha} \le x_2 \le kx_1^{1+\alpha} + h,\ -\sigma < x_1 < \sigma\}. \tag{4.12}$$

It is interesting that (4.11) holds even for $\omega \ne 0$.

4.2. Estimates on the Angular Velocity ω

Here we assume that S and Ω are both $C^{1,\alpha}$, $\alpha \in (0,1]$. As usual h is the distance of S to the boundary of Ω and let the points $P \in S$ and $Q \in \partial\Omega$ be two points that realize the distance h, that is $h = |\vec{PQ}|$. We choose a coordinate system x_1x_2 s.t. x_1 is tangent to $\partial\Omega$ at the point Q which coincides with the origin (i.e. $\boldsymbol{x}_Q = (0,0)$) and P has radius vector $\boldsymbol{x}_P = (0,h)$.

We assume that if $\boldsymbol{x} \in \bar{S}$ then

$$\boldsymbol{u}_S(\boldsymbol{x}) = \boldsymbol{u}_P + \omega(\boldsymbol{x} - \boldsymbol{x}_P)^{\perp}, \tag{4.13}$$

where $\boldsymbol{u}_P = (u_{P1},\ u_{P2})$ is the velocity of the solid at the point P. We denote by $\boldsymbol{\tau}_P$ the unit tangent vector to S at the point P. We have

Theorem 4.6. *Let $\Omega \subset I\!R^2$ and $S_0 \subset \Omega$ be uniformly $C^{1,\alpha}$, $\alpha \in (0,1]$ domains, S_0 being in addition bounded and*

$$\boldsymbol{u} \in L^\infty(0,T;(L^2(\Omega))^2) \cap L^1(0,T;W_0^{1,p}(S(t),\Omega)),\ p \in [1,\infty],\ \operatorname{div}\boldsymbol{u} = 0.$$

We assume that $h(t) < H$ for some $H > 0$. Then there exists a constant C depending only on S, Ω, H and p such that

$$|\omega| \le C h^{\frac{2\alpha}{p(1+\alpha)}\left(p-\frac{2+\alpha}{2\alpha}\right)} \|\nabla \boldsymbol{u}\|_{L^p(\Omega)}$$
$$|\boldsymbol{u}_P \cdot \boldsymbol{\tau}_P| \le C h^{\frac{1}{p}\left(p-\frac{2+\alpha}{1+\alpha}\right)} \|\nabla \boldsymbol{u}\|_{L^p(\Omega)}.$$

Proof. We consider the curves

$$\Gamma^+_{h,\rho} = \{(x_1,x_2):\ x_2 = h + k x_1^{1+\alpha},\ 0 \le x_1 \le \rho\},$$
$$\Gamma^-_{\rho} = \{(x_1,x_2):\ x_2 = -k x_1^{1+\alpha},\ 0 \le x_1 \le \rho\}.$$

From now on we choose a k and a ρ_0 such that $\Gamma^+_{h,\rho_0} \subset S$ and $\Gamma^-_{\rho_0} \subset I\!R^2 \backslash \bar{\Omega}$. We also denote

$$\mathcal{G}_{h,\rho} = \{(x_1,x_2):\ -k x_1^{1+\alpha} < x_2 < h + k x_1^{1+\alpha},\ -\rho < x_1 < \rho\},$$
$$\mathcal{G}^+_{h,\rho} = \{(x_1,x_2):\ -k x_1^{1+\alpha} < x_2 < h + k x_1^{1+\alpha},\ 0 < x_1 < \rho\}.$$

Let $0 < \tilde\rho < \rho \le \rho_0$. We apply divergence Theorem in the region $\mathcal{G}^+_{h,\rho} \backslash \mathcal{G}^+_{h,\tilde\rho}$ which is enclosed by the curves $x_1 = \rho$, $\Gamma^+_{h,\rho} \backslash \Gamma^+_{h,\tilde\rho}$, $x_1 = \tilde\rho$ and $\Gamma^-_{\rho} \backslash \Gamma^-_{\tilde\rho}$. Then we have (all line integrals are taken in the counter clockwise direction)

$$0 = \int_{\mathcal{G}^+_{h,\rho} \backslash \mathcal{G}^+_{h,\tilde\rho}} \operatorname{div} \boldsymbol{u}\, dx_1 dx_2 = \int_{-k\rho^{1+\alpha}}^{h+k\rho^{1+\alpha}} u_1(\rho,x_2)\, dx_2$$
$$+ \int_{\Gamma^+_{h,\rho}} \boldsymbol{u}\cdot\boldsymbol{n}\, ds - \int_{\Gamma^+_{h,\tilde\rho}} \boldsymbol{u}\cdot\boldsymbol{n}\, ds + \int_{-k\tilde\rho^{1+\alpha}}^{h+k\tilde\rho^{1+\alpha}} u_1(\tilde\rho,x_2)\, dx_2. \tag{4.14}$$

Straightforward calculations yields:

$$\int_{\Gamma^+_{h,\rho}} \boldsymbol{u}\cdot\boldsymbol{n}\, ds = \rho\, \boldsymbol{u}_P \cdot \boldsymbol{n}_P + k\rho^{1+\alpha}\, \boldsymbol{u}_P \cdot \boldsymbol{\tau}_P + \frac{\rho^2}{2}(1 + k^2\rho^{2\alpha})\omega; \tag{4.15}$$

where $\boldsymbol{n}_P = (0,-1)$ and $\boldsymbol{\tau}_P = (1,0)$. From (4.14) and (4.15) we get

$$\left|k(\rho^{1+\alpha} - \tilde\rho^{1+\alpha})\, \boldsymbol{u}_P\cdot\boldsymbol{\tau}_P + \frac{1}{2}\left[\rho^2 - \tilde\rho^2 + k^2(\rho^{2+2\alpha} - \tilde\rho^{2+2\alpha})\right]\omega\right|$$
$$\le |\rho - \tilde\rho|\, |\boldsymbol{u}_P\cdot\boldsymbol{n}_P| + \int_{-k\rho^{1+\alpha}}^{h+k\rho^{1+\alpha}} |u_1(\rho,x_2)|\, dx_2 + \int_{-k\tilde\rho^{1+\alpha}}^{h+k\tilde\rho^{1+\alpha}} |u_1(\tilde\rho,x_2)|\, dx_2$$
$$=: B.$$

By the same argument, this estimate is true even if $0 < \rho < \tilde\rho \le \rho_0$. We rewrite this as

$$-B \le k(\rho^{1+\alpha} - \tilde\rho^{1+\alpha})\, \boldsymbol{u}_P\cdot\boldsymbol{\tau}_P + \frac{1}{2}\left[\rho^2 - \tilde\rho^2 + k^2(\rho^{2+2\alpha} - \tilde\rho^{2+2\alpha})\right]\omega \le B. \tag{4.16}$$

We first integrate estimate (4.16) from $\rho = 0$ to $\rho = \sigma \le \rho_0$ for fixed $\tilde\rho$ and then we integrate from $\tilde\rho = 0$ to $\tilde\rho = \lambda\sigma$ for some $\lambda \in (0,1)$, to obtain

$$\left|\frac{k\lambda(1-\lambda^{1+\alpha})}{2+\alpha}\boldsymbol{u}_P\cdot\boldsymbol{\tau}_P\ \sigma^{3+\alpha} + \left(\frac{\lambda(1-\lambda^2)}{6} + \frac{k^2\lambda(1-\lambda^{2+2\alpha})}{3+2\alpha}\sigma^{2\alpha}\right)\sigma^4\omega\right|$$
$$\le \frac{1}{2}\lambda(1 - \lambda + \frac{2}{3}\lambda^2)|\boldsymbol{u}_P\cdot\boldsymbol{n}_P|\ \sigma^3 + \lambda\sigma\int_{\mathcal{G}^+_{h,\sigma}} |\boldsymbol{u}|\ dx_1dx_2 + \sigma\int_{\mathcal{G}^+_{h,\lambda\sigma}} |\boldsymbol{u}|\ dx_1dx_2. \tag{4.17}$$

We next estimate the terms in the right hand side. The first term in the right hand side has in fact be estimated in the proof of Theorem 4.1. More precisely, examination of the proof of Theorem 3.1 in [11] (see pp 321–322), for the 2D case shows that the following estimate holds true

$$|\boldsymbol{u}_P \cdot \boldsymbol{n}_P| \le C_1 \sigma^{-1-\frac{1}{p}} (h + 2k\sigma^{1+\alpha})^{2-\frac{1}{p}} \|\nabla \boldsymbol{u}\|_{L^p(\mathcal{G}_{h,\sigma})},$$

for a universal constant C_1. The second term of (4.17) is estimated, using Holder and then Poincare inequalities,

$$\int_{\mathcal{G}^+_{h,\sigma}} |\boldsymbol{u}|\; dx_1 dx_2 \le (h + 2k\sigma^{1+\alpha}) |\mathcal{G}^+_{h,\sigma}|^{\frac{p-1}{p}} \|\nabla \boldsymbol{u}\|_{L^p(\mathcal{G}^+_{h,\sigma})}.$$

Using also the fact that

$$|\mathcal{G}^+_{h,\sigma}| = \sigma \left(h + \frac{2k\sigma^{1+\alpha}}{2+\alpha} \right) < \sigma (h + 2k\sigma^{1+\alpha}),$$

we get

$$\int_{\mathcal{G}^+_{h,\sigma}} |\boldsymbol{u}|\; dx_1 dx_2 \le \sigma^{\frac{p-1}{p}} (h + 2k\sigma^{1+\alpha})^{2-\frac{1}{p}} \|\nabla \boldsymbol{u}\|_{L^p(\mathcal{G}^+_{h,\sigma})}.$$

We similarly estimate the third term in the right hand side, so that from (4.17) we have

$$\begin{aligned}
&\Big| \frac{k\lambda(1-\lambda^{1+\alpha})}{2+\alpha} \boldsymbol{u}_P \cdot \boldsymbol{\tau}_P \; \sigma^{3+\alpha} + \left(\frac{\lambda(1-\lambda^2)}{6} + \frac{k^2 \lambda (1-\lambda^{2+2\alpha})}{3+2\alpha} \sigma^{2\alpha} \right) \sigma^4 \omega \Big| \\
&\quad \le \lambda \sigma^{2-\frac{1}{p}} (h + 2k\sigma^{1+\alpha})^{2-\frac{1}{p}} \left[\frac{C_1}{2} (1 - \lambda + \frac{2}{3}\lambda^2) + 1 + \lambda^{-\frac{1}{p}} \right] \|\nabla \boldsymbol{u}\|_{L^p(\mathcal{G}_{h,\sigma})} \\
&\quad \le C_2 \lambda \sigma^{2-\frac{1}{p}} (h + 2k\sigma^{1+\alpha})^{2-\frac{1}{p}} \|\nabla \boldsymbol{u}\|_{L^p(\mathcal{G}_{h,\sigma})}.
\end{aligned}$$

In the last inequality we restrict $\lambda \in [\frac{1}{2}, \frac{3}{4}]$ and C_2 is a universal constant. After simplifying with λ and $\sigma^{3+\alpha}$ we get

$$\begin{aligned}
&\Big| \frac{k(1-\lambda^{1+\alpha})}{2+\alpha} \boldsymbol{u}_P \cdot \boldsymbol{\tau}_P + \left(\frac{(1-\lambda^2)}{6} + \frac{k^2 (1-\lambda^{2+2\alpha})}{3+2\alpha} \sigma^{2\alpha} \right) \sigma^{1-\alpha} \omega \Big| \\
&\quad \le C_2 \sigma^{-1-\alpha-\frac{1}{p}} (h + 2k\sigma^{1+\alpha})^{2-\frac{1}{p}} \|\nabla \boldsymbol{u}\|_{L^p(\mathcal{G}_{h,\sigma})}.
\end{aligned}$$

Since $\lambda,\ k$ can take many different values we easily conclude that each of the terms $|\boldsymbol{u}_P \cdot \boldsymbol{\tau}_P|$ and $|\sigma^{1-\alpha}\omega|$ is separately bounded by a suitable multiple of the right hand side and therefore we arrive at

$$\begin{aligned}
|\boldsymbol{u}_P \cdot \boldsymbol{\tau}_P| &\le C_3 \sigma^{-1-\alpha-\frac{1}{p}} (h + 2k\sigma^{1+\alpha})^{2-\frac{1}{p}} \|\nabla \boldsymbol{u}\|_{L^p(\mathcal{G}_{h,\sigma})} \\
|\omega| &\le C_3 \sigma^{-2-\frac{1}{p}} (h + 2k\sigma^{1+\alpha})^{2-\frac{1}{p}} \|\nabla \boldsymbol{u}\|_{L^p(\mathcal{G}_{h,\sigma})},
\end{aligned}$$

with a positive constant C_3 depending on k, ρ_0. To conclude the proof we finally choose $\sigma = \left(\frac{h}{H}\right)^{\frac{1}{1+\alpha}} \rho_0 \le \rho_0$. □

Remark 4.4. It follows from the proof that in the statement of Theorem 4.6 we can replace the norm $\|\nabla \boldsymbol{u}\|_{L^p(\Omega)}$ by $\|\nabla \boldsymbol{u}\|_{L^p(\mathcal{G}_{h,\sigma})}$. Also, the dependence of the constant C on S, Ω is through k, ρ_0 and α.

Remark 4.5. We note that Theorems 4.3 and 4.6 are in agreement with Theorem 3.2 of the static case. In fact, one can easily obtain Theorem 3.2 from Theorems 4.3 and 4.6 through a limiting process $h \to 0$ taking into account Remark 3.1 and (4.8).

5. Examples and Optimality of the Results

In this section we first present an example, motivated from the motion of a rigid body in an incompressible fluid, which shows the possibility of non zero collision speed. The other examples of the section demonstrate the optimality of the results obtained in Sect. 4.

Throughout this section $\Omega = I\!R^2_+ = \{(x_1, x_2) : x_2 > 0\}$ and $0 \le h = h(t) < H$ is a smooth function of time. We next define

$$S(t) = \{(x_1, x_2) : \ x_2 \ge k|x_1|^{1+\alpha} + h(t),\ x_1^2 + (x_2 - h(t))^2 \le \rho_0\}. \tag{5.1}$$

Clearly $h(t) = \operatorname{dist}(S(t), \partial\Omega)$. Let $\boldsymbol{a} = (0, \dot h(t))$ and

$$\Phi(\boldsymbol{x}, t) = \boldsymbol{a} \cdot \boldsymbol{x}^\perp = -\dot h(t) x_1.$$

Let ϕ, ψ be the cutoff functions defined in (2.7) with $\rho = H + \rho_0$. We consider the vector field

$$\boldsymbol{u}(\boldsymbol{x}, t) = \nabla^\perp \Psi(\boldsymbol{x}, t) := \nabla^\perp \left(\psi(|\boldsymbol{x}|) \phi \left(\frac{x_2}{k x_1^{1+\alpha} + h(t)} \right) \Phi(\boldsymbol{x}, t) \right) . \tag{5.2}$$

One easily sees that

$$\begin{aligned} \boldsymbol{u}(\boldsymbol{x}, t) &= (0, \dot h(t)),\ \boldsymbol{x} \in S(t), \\ \operatorname{div}\boldsymbol{u}(\boldsymbol{x}, t) &= 0,\ x \in I\!R^2_+,\ t > 0, \\ u((x_1, 0), t) &= 0. \end{aligned}$$

Since $\boldsymbol{u}$ is zero outside $|\boldsymbol{x}| < 2\rho$ we can obviously consider bounded domains Ω as well. We then have

Example 1. (Non zero collision speed) We interpret $\boldsymbol{u}$ in (5.2) as the velocity field of a viscous fluid in $I\!R^2_+$ and $S(t)$ is a rigid body moving in the fluid, see e.g [6,11]. It is then natural to require

$$\boldsymbol{u} \in L^\infty(0, T; (L^2(\Omega))^2) \cap L^2(0, T; W^{1,2}_0(S(t), \Omega)). \tag{5.3}$$

Using Lemma (7.1) of the “Appendix” this is easily seen to be true provided that

$$\begin{aligned} &\dot h \in L^\infty(0, T), \quad \dot h^2 h^{-\frac{3\alpha}{1+\alpha}} \in L^1(0, T),\ \alpha \in (0, 1], \\ &\dot h \in L^\infty(0, T), \quad \dot h^2 |\ln h| \in L^1(0, T),\ \alpha = 0. \end{aligned}$$

For $\alpha = 1$ the above conditions are the same as in [11], Lemmas 4.1, 4.2 where a completely different example is constructed.

We also note that $\ddot h \in L^2(0, T)$ implies $\Psi_t \in L^2(0, T; L^2(\Omega))$ which, by means of (5.2) has as a consequence that $\boldsymbol{u}_t \in L^2(0, T; H^{-1}(\Omega))$. Then, the pair $(\boldsymbol{u}(\boldsymbol{x}, t), S(t))$, according to [11], is a weak solution of the problem of motion of a rigid body in a viscous fluid, with a forcing term $\boldsymbol{g} \in L^2(0, T; H^{-1}(\Omega))$.

It is shown in [11], p. 323 that for $\alpha \ge 1/2$ a body comes to the external boundary with zero speed. We will show that for $0 \le \alpha < 1/2$ the body hits the boundary with a non zero speed. Indeed, let

$$h(t) = (T - t)^\theta,\ \theta \ge 1, \tag{5.4}$$

condition (5.3) is satisfied for

$$\theta \ge 1 \text{ if } \alpha \in [0, 1/2) \text{ or } \theta > \frac{1+\alpha}{2-\alpha} \ge 1 \text{ if } \alpha \in [1/2, 1] .$$

Moreover $\ddot h \in L^2(0, T)$ provided that $\theta = 1$ or $\theta > 3/2$. Thus, for $0 \le \alpha < 1/2$ we can take $h(t) = T - t$ which gives a non zero collision speed ($\dot h = -1$), whereas for $\alpha \ge 1/2$ the collision speed is indeed zero.

Example 2. (On the optimality of Theorems 4.1 and 4.3) Suppose that $\alpha = 0$. From the estimate (4.10) of Remark 4.3 we have that

$$C|\dot h|^p h^{2-p} \le \int_{\Pi_h} |\nabla \boldsymbol{u}|^p dx_1 dx_2,$$

whereas from (7.3) of "Appendix" we see that

$$\int_{\Pi_h} |\nabla \boldsymbol{u}|^p dx_1 dx_2 \leq C|\dot{h}|^p h^{2-p}.$$

Therefore estimate (4.10) of Remark 4.3 is optimal for any $p \geq 1$. Similarly by comparing estimate (4.3) of Theorem 4.3 and (7.2) (with $\alpha = 0$), we conclude that the estimate of Theorem 4.3 is optimal for $p > 2$.

Consider now the case $\alpha > 0$. From estimate (4.11) of Remark 4.2 we have that

$$C|\dot{h}|^p h^{-\beta p} \leq \int_{\mathcal{G}_{h,\sigma_0}} |\nabla \boldsymbol{u}|^p dx_1 dx_2,$$

whereas using the asymptotics of (7.3) we see that

$$\int_{\mathcal{G}_{h,\sigma_0}} |\nabla \boldsymbol{u}|^p dx_1 dx_2 \leq C|\dot{h}|^p h^{-\beta p}.$$

Therefore (4.11) is optimal for any $p \geq 1$. A similar argument shows that estimate (4.1) is optimal for $p > \frac{2+\alpha}{1+2\alpha}$.

Example 3. (On the optimality of Theorem 4.6) For simplicity we consider the case $\alpha = 1$. We also assume that $p > \frac{3}{2}$. This time S is the disc of radius r centered at $\boldsymbol{x}_* = (0, h+r)$ and h is taken to be a constant. The disc is rotated with angular velocity ω about its center. More precisely

$$\Phi(\boldsymbol{x}) = \frac{1}{2}\omega|\boldsymbol{x}|^2 - \omega(r+h)x_2,$$

so that $\nabla^\perp \Phi = \omega(\boldsymbol{x} - \boldsymbol{x}_*)^\perp$ and the vector field is now given by

$$\boldsymbol{u} = \nabla^\perp \left(\psi(|\boldsymbol{x}|)\phi\left(\frac{x_2}{h + \frac{1}{2r}x_1^2}\right)\Phi(\boldsymbol{x})\right),$$

here ϕ, ψ are the usual cutoff functions defined in (2.7) with $\rho = h + 2r$. One easily sees that

$$\begin{aligned} \boldsymbol{u}(\boldsymbol{x}, t) &= \omega(\boldsymbol{x} - \boldsymbol{x}_*)^\perp,\ \boldsymbol{x} \in S(t), \\ \mathrm{div}\boldsymbol{u}(\boldsymbol{x}, t) &= 0,\ x \in I\!R^2_+,\ t > 0, \\ u(x_1, 0) &= 0. \end{aligned}$$

From Theorem (4.6) we have that (for $\alpha = 1$),

$$C|\omega|^p h^{\left(\frac{3}{2}-p\right)} \leq \int_\Omega |\nabla \boldsymbol{u}|^p dx_1 dx_2,$$

whereas using the estimates $|\Phi(x)| \leq C|\omega|(x_1^2 + x_2)$ and $|\nabla\Phi(x)| < C|\omega|$ and working as in the in the "Appendix" we get

$$\int_\Omega |\nabla \boldsymbol{u}|^p dx_1 dx_2 \leq C|\omega|^p h^{\left(\frac{3}{2}-p\right)}.$$

As a consequence the estimate for $|\omega|$ is optimal. The estimate for the tangential part can be treated similarly.

Example 4. (A non solenoidal vector field) For ϕ, ψ as in (2.7) with $\rho = H + \rho_0$ and $S(t)$, Ω as in (5.1) we define

$$\boldsymbol{u}(\boldsymbol{x}, t) = \psi(|\boldsymbol{x}|)\phi\left(\frac{x_2}{kx_1^{1+\alpha} + h(t)}\right)\boldsymbol{u}_S,\ \boldsymbol{u}_S = (0, \dot{h}). \tag{5.5}$$

We now have that $\boldsymbol{u}(\boldsymbol{x}, t) = (0, \dot{h}(t))$ for $\boldsymbol{x} \in S(t)$, $u(x_1, 0) = 0$ but $\boldsymbol{u}$ is not necessarily divergence free. From estimate (4.1) we have that

$$C|\dot{h}|^p h^{-\beta p} \leq \int_\Omega |\nabla \boldsymbol{u}|^p dx_1 dx_2, \tag{5.6}$$

On the other hand, similar to previous examples calculations, yield

$$\int_{I\!R_+^2} |\nabla \boldsymbol{u}|^p dx_1 dx_2 \leq \begin{cases} C|\dot{h}|^p, & p < \frac{2+\alpha}{1+\alpha}, \\ C|\dot{h}|^{\frac{2+\alpha}{1+\alpha}} \, |\ln h|, & p = \frac{2+\alpha}{1+\alpha}, \\ C|\dot{h}|^p |h|^{-\left(p - \frac{2+\alpha}{1+\alpha}\right)}, & p > \frac{2+\alpha}{1+\alpha} . \end{cases}$$

One easily sees that for $p > \frac{2+\alpha}{1+\alpha}$ one reaches a contradiction when $\alpha > 0$. Thus estimate (4.1) fails for the non solenoidal case for $\alpha > 0$, as opposed to the case $\alpha = 0$ of Theorem 4.3.

6. A Rotating Body Approaching the Boundary: A Non Collision Result

In this section we present an application of Theorem 4.6.

Let $S(t) \subset \Omega \subset I\!R^2$ be two smooth domains (e.g. C^2). The rigid body $S(t)$ is bounded, it has density one and its center of mass has radius vector $\boldsymbol{x}_c(t)$. We denote by $\mathcal{F}(t)$ the fluid region, that is $\mathcal{F}(t) = \Omega \backslash S(t)$. We assume that there exists a vector field $\boldsymbol{u}(\boldsymbol{x}, t)$

$$\boldsymbol{u} \in L^\infty(0, T; (L^2(\Omega))^2) \cap L^2(0, T; W_0^{1,2}(S(t), \Omega)), \tag{6.1}$$

such that

$$\boldsymbol{u}|_{S(t)} = \boldsymbol{a}(t) + \omega(t)(\boldsymbol{x} - \boldsymbol{x}_c(t))^\perp, \; \boldsymbol{x} \in S(t), \tag{6.2}$$

where $\boldsymbol{a}$ and ω are in $H^1(0, \tilde{T})$ for any $\tilde{T} \in (0, T)$. The assumption on $\boldsymbol{a}$ and ω is reasonable in view of Theorem 2.2 in [13]. Moreover $\boldsymbol{u}(\boldsymbol{x}, t)$ satisfies the stationary Stokes system in $\mathcal{F}(t)$ with Dirichlet boundary conditions. More precisely, in addition to (6.2), for almost all t, $\boldsymbol{u}$ satisfies

$$\begin{aligned} \operatorname{div} \mathbb{T} &= 0, \; \mathbb{T} := 2D(\boldsymbol{u}) - p\mathbb{I}, \; x \in \mathcal{F}(t), \\ \operatorname{div} \boldsymbol{u} &= 0, \\ \boldsymbol{u}|_{\partial S(t)} &= \boldsymbol{a}(t) + \omega(t)(\boldsymbol{x} - \boldsymbol{x}_c(t))^\perp, \\ \boldsymbol{u}|_{\partial\Omega} &= 0. \end{aligned} \tag{6.3}$$

Here p is the pressure function and D is the deformation tensor given in (1.2). As usual $h(t)$ is the distance of $S(t)$ to $\partial\Omega$. In view of Lemma 4.5 and (6.2) we have

$$\dot{h} = -(\boldsymbol{a} + \omega(\boldsymbol{x}_P - \boldsymbol{x}_c(t))^\perp) \cdot \boldsymbol{n}_P; \tag{6.4}$$

here P is a point that realizes the distance, $\boldsymbol{x}_P$ the radius vector of P and $\boldsymbol{n}_P$ the outward to S normal vector.

At each time t the fluid exerts on the body a force $\boldsymbol{F}$ and a torque N. By Newton's balance equations we have that ($|S|$ is the area of S)

$$\begin{aligned} |S| \, \dot{\boldsymbol{a}} = -\boldsymbol{F} &= -\int_{\partial S} (2D(\boldsymbol{u})\boldsymbol{n} - p\mathbb{I}\boldsymbol{n}) dS \\ J\dot{\omega} = -N &= -\int_{\partial S} (2D(\boldsymbol{u})\boldsymbol{n} - p\mathbb{I}\boldsymbol{n}) \cdot (\boldsymbol{x} - \boldsymbol{x}_c)^\perp dS, \; J = \int_S |\boldsymbol{x} - \boldsymbol{x}_c|^2 d\boldsymbol{x}; \end{aligned} \tag{6.5}$$

Here $\boldsymbol{n}$ is the outward to $\mathcal{F}(t)$ normal vector.

We then have

Theorem 6.1. *Assume that the functions $\boldsymbol{u}$, h, $\boldsymbol{a}$, ω satisfy (6.1)–(6.5). If $h(0)$ is positive then $h(t)$ remains positive in $[0, T]$ and therefore the rigid body S never touches the boundary of Ω.*

The rest of the section is devoted to the proof of this Theorem. We first prove an auxiliary Lemma.

Lemma 6.2. *Let* $\boldsymbol{u}$ *be as in Theorem 6.1. Then for a.a. t*

$$\int_{\mathcal{F}} \nabla \boldsymbol{u} \cdot (\nabla \boldsymbol{u})^T d\boldsymbol{x} = 2\omega^2 |S| = \int_S |\nabla \boldsymbol{u}|^2 d\boldsymbol{x}. \tag{6.6}$$

$$\boldsymbol{a} \cdot \boldsymbol{F} + \omega N = \int_\Omega |\nabla \boldsymbol{u}|^2 d\boldsymbol{x}. \tag{6.7}$$

Proof. (**i**) We note that in S we have $D(\boldsymbol{u}) = 0$, $|\nabla \boldsymbol{u}| = \sqrt{2}|\omega|$ and $\nabla \boldsymbol{u} \cdot (\nabla \boldsymbol{u})^T = -2\omega^2$. Since $\boldsymbol{u}$ is divergence free we have

$$\begin{aligned}\int_\Omega \nabla \boldsymbol{u} \cdot (\nabla \boldsymbol{u})^T dx &= \int_\Omega \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i} dx = \int_\Omega \frac{\partial}{\partial x_j}\left(u_i \frac{\partial u_j}{\partial x_i}\right) dx \\ &= \int_\Omega \operatorname{div}(\boldsymbol{u} \cdot \nabla \boldsymbol{u}) dx = \int_{\partial\Omega} (\boldsymbol{u} \cdot \nabla \boldsymbol{u}) \cdot \boldsymbol{n} ds = 0.\end{aligned}$$

Consequently

$$\begin{aligned}0 &= \int_\Omega \nabla \boldsymbol{u} \cdot (\nabla \boldsymbol{u})^T dx = \int_{\mathcal{F}} \nabla \boldsymbol{u} \cdot (\nabla \boldsymbol{u})^T dx + \int_S \nabla \boldsymbol{u} \cdot (\nabla \boldsymbol{u})^T dx \\ &= \int_{\mathcal{F}} \nabla \boldsymbol{u} \cdot (\nabla \boldsymbol{u})^T dx - 2\omega^2 |S|,\end{aligned}$$

from which (6.6) follows.

(**ii**) We multiply the first equation of (6.3) by $\boldsymbol{u}$ and integrate by parts in $\mathcal{F}$ to get

$$\begin{aligned}0 &= \int_{\mathcal{F}} \boldsymbol{u} \cdot \operatorname{div} \mathbb{T}\, dx = -\int_{\mathcal{F}} \nabla \boldsymbol{u} \cdot \mathbb{T} dx + \int_{\partial\mathcal{F}} \boldsymbol{u} \cdot \mathbb{T}\boldsymbol{n}\, ds \\ &= -2\int_{\mathcal{F}} \nabla \boldsymbol{u} \cdot D(\boldsymbol{u}) dx + \int_{\partial S} (\boldsymbol{a} + \omega(\boldsymbol{x} - \boldsymbol{x}_c)^\perp) \cdot (2D(\boldsymbol{u})\boldsymbol{n} - p\mathbb{I}\boldsymbol{n}) ds \\ &= -\int_{\mathcal{F}} |\nabla \boldsymbol{u}|^2 dx - 2\omega^2 |S| + \boldsymbol{a} \cdot \int_{\partial S} (2D(\boldsymbol{u})\boldsymbol{n} - p\mathbb{I}\boldsymbol{n}) ds \\ &\quad + \omega \int_{\partial S} (2D(\boldsymbol{u})\boldsymbol{n} - p\mathbb{I}\boldsymbol{n}) \cdot (\boldsymbol{x} - \boldsymbol{x}_c)^\perp ds \\ &= -\int_\Omega |\nabla \boldsymbol{u}|^2 dx + \boldsymbol{a} \cdot \boldsymbol{F} + \omega N.\end{aligned}$$

In the above calculation we also used (6.5) and (6.6). This shows (6.7) and completes the proof of the Lemma. □

From (6.5) and (6.7) we have

$$\frac{1}{2}\frac{d}{dt}\left(|S||\boldsymbol{a}|^2 + J\omega^2\right) = -\int_\Omega |\nabla \boldsymbol{u}|^2 dx. \tag{6.8}$$

We set

$$k^2(t) := |S|\, |\boldsymbol{a}|^2 + J\omega^2,$$

so that (6.8) reads

$$k(t)\dot{k}(t) = -\|\nabla \boldsymbol{u}\|^2_{L^2(\Omega)}. \tag{6.9}$$

Let us recall our notation from the previous sections. At each time t we denote by $P \in \partial S$ a point that realizes the distance $h(t)$. We denote by $\boldsymbol{u}_P$ the velocity at the point P and by $\boldsymbol{n}_P$ and $\boldsymbol{\tau}_P$ the external (to S) normal and tangential unit vectors, respectively, at the point P. We then have

Lemma 6.3. *There are two positive constants c, C depending only on S such that for any t*

$$c(|\dot{h}| + |\boldsymbol{u}_P \cdot \boldsymbol{\tau}_P| + |\omega|) \le k(t) \le C(|\dot{h}| + |\boldsymbol{u}_P \cdot \boldsymbol{\tau}_P| + |\omega|). \tag{6.10}$$

Proof. We choose coordinate system such that the x_1 axis is parallel to $\boldsymbol{\tau}_P$. Let $\boldsymbol{x}_P = (x_{P1},\ x_{P2})$ be the radius vector of P. We then have

$$\begin{aligned}\boldsymbol{u}_P &= (\boldsymbol{u}_P \cdot \boldsymbol{\tau}_P,\ \boldsymbol{u}_P \cdot \boldsymbol{n}_P) = \boldsymbol{a} + \omega(\boldsymbol{x}_P - \boldsymbol{x}_c)^{\perp} \\ &= (a_1 + \omega(x_{P2} - x_{c2}),\ a_2 - \omega(x_{P1} - x_{c1}),\end{aligned}$$

whence,

$$\begin{aligned}a_1 &= \boldsymbol{u}_P \cdot \boldsymbol{\tau}_P - \omega(x_{P2} - x_{c2}) \\ a_2 &= \boldsymbol{u}_P \cdot \boldsymbol{n}_P + \omega(x_{P1} - x_{c1}).\end{aligned}$$

Thus,

$$k^2(t) = |S|((\boldsymbol{u}_P \cdot \boldsymbol{\tau}_P - \omega(x_{P2} - x_{c2}))^2 + (\boldsymbol{u}_P \cdot \boldsymbol{n}_P + \omega(x_{P1} - x_{c1}))^2) + J\omega^2.$$

The upper estimate follows easily taking into account that, by Lemma 4.5, $\dot{h} = -\boldsymbol{u}_P \cdot \boldsymbol{n}_P$.

For the lower estimate, expanding the squares and using Young's inequality we obtain that for any $\varepsilon > 0$ we have

$$\begin{aligned}k^2(t) \geq\ & |S|(1-\varepsilon)(|\boldsymbol{u}_P \cdot \boldsymbol{\tau}_P|^2 + |\boldsymbol{u}_P \cdot \boldsymbol{n}_P|^2) \\ & + \left(J + |\boldsymbol{x}_P - \boldsymbol{x}_c|^2|S| - \frac{|\boldsymbol{x}_P - \boldsymbol{x}_c|^2|S|}{\varepsilon}\right)\omega^2.\end{aligned} \tag{6.11}$$

Choosing

$$1 - \frac{J}{J + |S|\ \max_{x\in\partial S}|\boldsymbol{x} - \boldsymbol{x}_c|^2} < \varepsilon < 1,$$

all coefficients in (6.11) are strictly positive and the lower bound follows. □

Using (6.10) and then Theorems 4.1 and 4.6 (for $\alpha = 1$ and $p = 2$), we have

$$\begin{aligned}k(t) &\leq C(|\dot{h}| + |\boldsymbol{u}_P \cdot \boldsymbol{\tau}_P| + |\omega|) \\ &\leq c(h^{\frac{1}{4}} + h^{\frac{3}{4}})\|\nabla \boldsymbol{u}\|_{L^2(\Omega)} \\ &\leq ch^{\frac{1}{4}}\|\nabla \boldsymbol{u}\|_{L^2(\Omega)}.\end{aligned} \tag{6.12}$$

From (6.9) and (6.12) we have that

$$k\dot{k} \leq -c_1 k^2 h^{-\frac{1}{2}}. \tag{6.13}$$

Assuming that h is positive in $(0, t_*)$ and $h(t_*) = 0$ at a time $t_* \leq T$ we will reach a contradiction. Since $k(t)$ is non increasing we also have $k(t) > 0$ in $(0, t_*)$. (If it becomes zero then the body "freezes" and never touches the boundary of Ω). We rewrite (6.13) as

$$\frac{d}{dt}(\ln k(t)) \leq -c_1 h^{-\frac{1}{2}}(t),\ t \in (0, t_*). \tag{6.14}$$

It is a consequence of Corollary 4.2, with $\alpha = 1$, $p = 2$ (hence $\beta = 3/4$) and $q = 2$, that there exists a positive function $\varepsilon(t)$ such that

$$h(t) = \varepsilon^2(t)(t_* - t)^2,\ t \in (0, t_*),\ \varepsilon(t) \xrightarrow[t\uparrow t_*]{} 0. \tag{6.15}$$

It follows that given any $\varepsilon_0 > 0$ there exists a $t_0 \in (0, t_*)$ such that

$$\varepsilon(t) < \varepsilon_0,\ t \in (t_0, t_*). \tag{6.16}$$

Integrating (6.14) in (t_0, t) with $t < t_*$ we get

$$k(t) \leq k(t_0)e^{-c_1\int_{t_0}^{t} h^{-\frac{1}{2}}(s)}ds,\ t \in (t_0, t_*).$$

Taking into account (6.10) we note that

$$h(t) = -\int_t^{t_*} \dot{h}(s)ds \leq c_2 \int_t^{t_*} k(s)ds.$$

Combining the previous two estimates we have that

$$h(t) \leq c_3 \int_t^{t_*} e^{-c_1 \int_{t_0}^s h^{-\frac{1}{2}}(\tau)d\tau} ds,\ t \in (t_0, t_*), \tag{6.17}$$

with $c_3 = c_2 k(t_0) > 0$. To reach a contradiction we show

Lemma 6.4. *There is no function h satisfying both (6.15) and (6.17).*

Proof. We set

$$F(t) := \int_t^{t_*} e^{-c_1 \int_{t_0}^s h^{-\frac{1}{2}}(\tau)d\tau} ds > 0,\ t_0 < t < t_*.$$

We may think of (6.17) as giving a lower bound for F, whereas using (6.15) we can obtain an upper bound for F. It turns out that the two are incompatible.

We first note that for $t \in (t_0, t_*)$,

$$\begin{aligned} F'(t) &= -e^{-c_1 \int_{t_0}^t h^{-\frac{1}{2}}(\tau)d\tau} < 0, \\ F''(t) &= -c_1 h^{-\frac{1}{2}}(t) F'(t) > 0. \end{aligned} \tag{6.18}$$

In particular,

$$h(t) = \left(\frac{c_1 F'}{F''} \right)^2,$$

and (6.17) can be rewritten as

$$\left(\frac{c_1 F'}{F''} \right)^2 \leq c_3 F, \text{ or } \frac{F''}{c_1 F'} \geq \frac{F^{-\frac{1}{2}}}{\sqrt{c_3}}.$$

The last inequality is easily seen to be equivalent to

$$\left(\sqrt{c_3}\ F' + 2c_1 F^{\frac{1}{2}} \right)' \geq 0,\ t \in (t_0, t_*). \tag{6.19}$$

Taking into account (6.15), (6.16) and (6.18) we easily arrive at

$$|F'(t)| \leq (t_* - t_0)^{-\frac{c_1}{\varepsilon_0}} (t_* - t)^{\frac{c_1}{\varepsilon_0}},\ t \in (t_0, t_*),$$

which upon integration from t to t_* gives

$$0 < F(t) \leq \frac{(t_* - t_0)^{-\frac{c_1}{\varepsilon_0}}}{\frac{c_1}{\varepsilon_0} + 1} (t_* - t)^{\frac{c_1}{\varepsilon_0} + 1},\ t \in (t_0, t_*). \tag{6.20}$$

A direct consequence of the above two estimates is that

$$F(t_*) = F'(t_*) = 0.$$

Integrating (6.19) from t to t_* we get

$$\sqrt{c_3}\ F' + 2c_1 F^{\frac{1}{2}} \leq 0,$$

or

$$\frac{1}{2}\sqrt{c_3} F_3' F^{-\frac{1}{2}} + c_1 \leq 0,$$

which can be rewritten as

$$(\sqrt{c_3}\ F^{\frac{1}{2}} - c_1(t_* - t))' \leq 0\ t \in (t_0, t_*).$$

Integrating once more from t to t_* we end up with

$$F(t) \geq \frac{c_1^2}{c_3}(t_* - t)^2,\ t \in (t_0, t_*).$$

Since ε_0 can be chosen arbitrarily small, the last estimate contradicts (6.20). □

Compliance with ethical standards
Conflict of interest The authors declare that they have no conflict of interest.

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7. Appendix

We recall that

$$\beta = \frac{1+2\alpha}{p(1+\alpha)}\left(p - \frac{2+\alpha}{1+2\alpha}\right).$$

We then have

Lemma 7.1. *Let $p \geq 1$ and $0 \leq \alpha \leq 1$. Then, for $\boldsymbol{u}$ as defined in (5.2) there holds*

$$\int_{I\!R^2_+} |\boldsymbol{u}|^p dx_1 dx_2 \leq \begin{cases} C|\dot{h}|^p, & \alpha p < 2+\alpha\ , \\ C|\dot{h}|^p\,|\ln h|, & \alpha p = 2+\alpha, \\ C|\dot{h}|^p h^{\frac{2+\alpha-\alpha p}{1+\alpha}}, & \alpha p > 2+\alpha, \end{cases} \tag{7.1}$$

$$\int_{I\!R^2_+} |\nabla \boldsymbol{u}|^p dx_1 dx_2 \leq \begin{cases} C|\dot{h}|^p, & p < \frac{2+\alpha}{1+2\alpha}, \\ C|\dot{h}|^{\frac{2+\alpha}{1+2\alpha}}\,|\ln h|, & p = \frac{2+\alpha}{1+2\alpha}, \\ C|\dot{h}|^p h^{-\beta p}, & p > \frac{2+\alpha}{1+2\alpha}\ . \end{cases} \tag{7.2}$$

In addition, the following local estimates hold true for $p \geq 1$

$$\int_{\mathcal{G}_{h,\sigma_0}} |\nabla \boldsymbol{u}|^p dx_1 dx_2 \leq C|\dot{h}|^p h^{-\beta p},\ 0 < \alpha \leq 1, \tag{7.3}$$

$$\int_{\Pi_h} |\nabla \boldsymbol{u}|^p dx_1 dx_2 \leq \quad C|\dot{h}|^p h^{2-p},\ \alpha = 0, \tag{7.4}$$

where $\mathcal{G}_{h,\sigma_0}$ is defined in (4.12) and Π_h in the proof of Theorem 4.3.

Proof. For

$$\phi = \phi\left(\frac{x_2}{kx_1^{1+\alpha} + h(t)}\right),$$

we have that

$$\begin{aligned} \nabla \boldsymbol{u} &= \Phi\nabla\nabla^{\perp}\phi + \nabla\Phi\nabla^{\perp}\phi + \nabla\phi\nabla^{\perp}\Phi\ + \cdots \\ &=: I_1 + I_2 + I_3 + \cdots \end{aligned} \tag{7.5}$$

A straightforward calculation shows that for $i, j = 1, 2$

$$\frac{\partial \phi}{\partial x_i} = 0,\ x_2 \geq kx_1^{1+\alpha} + h,$$

$$\left|\frac{\partial \phi}{\partial x_i}\right| \leq \frac{C}{kx_1^{1+\alpha} + h},\ 0 \leq x_2 \leq kx_1^{1+\alpha} + h,$$

$$\left|\frac{\partial^2 \phi}{\partial x_i x_j}\right| \leq \frac{C(1 + \alpha h x_1^{\alpha-1})}{(kx_1^{1+\alpha} + h)^2},\ 0 \leq x_2 \leq kx_1^{1+\alpha} + h.$$

We then compute, for $\gamma > 0$ small enough but fixed

$$
\begin{aligned}
I_1 &= \int_{I\!R^2_+} |\Phi|^p |\nabla\nabla^\perp \phi|^p dx_1 dx_2 \\
&\leq 2\int_0^\gamma \int_0^{kx_1^{1+\alpha}+h} |\Phi|^p |\nabla\nabla^\perp \phi|^p dx_2 dx_1 + 2\int_\gamma^{2\rho} \int_0^{kx_1^{1+\alpha}+h} |\Phi|^p |\nabla\nabla^\perp \phi|^p dx_2 dx_1 \\
&\leq C|\dot h|^p \left(\int_0^\gamma \int_0^{kx_1^{1+\alpha}+h} \frac{x_1^p\, dx_2 dx_1}{(kx_1^{1+\alpha}+h)^{2p}} + \int_0^\gamma \int_0^{kx_1^{1+\alpha}+h} \frac{\alpha^p x_1^{\alpha p}\, dx_2 dx_1}{(kx_1^{1+\alpha}+h)^{p}} + O_h(1) \right) \\
&\leq C|\dot h|^p \left(\int_0^\gamma \frac{x_1^p\, dx_1}{(kx_1^{1+\alpha}+h)^{2p-1}} + \alpha^p \int_0^\gamma \frac{x_1^{\alpha p}\, dx_1}{(kx_1^{1+\alpha}+h)^{p-1}} + O_h(1) \right)
\end{aligned}
$$

To estimate the integrals above we use the fact that for $0 \leq \alpha \leq 1$ and $b, q \in I\!R$ we have

$$
\int_0^\gamma \frac{x_1^b\, dx_1}{(kx_1^{1+\alpha}+h)^q} = \frac{h^{\frac{b-\alpha}{1+\alpha}-q+1}}{1+\alpha} \int_0^{\frac{\gamma^{1+\alpha}}{h}} \frac{z^{\frac{b-\alpha}{1+\alpha}}\, dz}{(kz+1)^q}, \quad \left(z = \frac{x^{1+\alpha}}{h} \right).
$$

We then get after elementary manipulations

$$
I_1 \leq \begin{cases} C|\dot h|^p, & p < \frac{2+\alpha}{1+2\alpha}, \\ C|\dot h|^{\frac{2+\alpha}{1+2\alpha}} |\ln h|, & p = \frac{2+\alpha}{1+2\alpha}, \\ C|\dot h|^p |h|^{-\frac{1+2\alpha}{1+\alpha}\left(p - \frac{2+\alpha}{1+2\alpha}\right)}, & p > \frac{2+\alpha}{1+2\alpha}\ . \end{cases}
$$

We similarly calculate for $i = 2, 3$

$$
I_i \leq \begin{cases} C|\dot h|^p, & p < \frac{2+\alpha}{1+\alpha}, \\ C|\dot h|^{\frac{2+\alpha}{1+\alpha}} |\ln h|, & p = \frac{2+\alpha}{1+\alpha}, \\ C|\dot h|^p |h|^{-\left(p - \frac{2+\alpha}{1+\alpha}\right)}, & p > \frac{2+\alpha}{1+\alpha}\ . \end{cases}
$$

Combining the estimates of I_i, $i = 1, 2, 3$ and noticing that the omitted terms in (7.5) are not as important we conclude (7.2). Estimate (7.1) is similar and simpler. □

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