Best constants for weighted Hardy inequalities in the exterior of balls and circular cylinders

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Abstract

We consider the weighted Hardy inequality

$$\int_\Omega \frac{|\nabla u(x)|^2}{d^{s-2}(x)}\,dx \ge c_s(\Omega)\int_\Omega \frac{u^2(x)}{d^s(x)}\,dx, \qquad \forall u \in C_c^\infty(\Omega).$$

For $s \ge 1$, $n \ge 2$, $s \ne n$ we compute the best constant in the case where Ω is either the complement of a ball or the complement of a circular cylinder. In both cases the domains are not weakly mean convex.

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1 Introduction and main result

The classical Hardy inequality involving the distance to the boundary states that for a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, $n \ge 2$ there exists a positive constant c_Ω such that

$$\int_\Omega |\nabla u(x)|^2\,dx \ge c_\Omega \int_\Omega \frac{u^2(x)}{d^2(x)}\,dx, \qquad \forall u \in C_c^\infty(\Omega), \tag{1.1}$$

where $d(x) = \mathrm{dist}(x, \partial\Omega)$.

In general the best constant c_Ω depends on Ω. However under the assumption of convexity of Ω or even weak mean convexity, that is,

$$-\Delta d(x) \ge 0, \qquad x \in \Omega,$$

in the distributional sense, one can establish that $c_\Omega = \frac{1}{4}$, under very mild regularity assumptions on the boundary of Ω. We emphasize that in this case Ω can be unbounded.

There are very few examples of non weakly mean convex domains where one can identify the Hardy constant c_Ω. In the two dimensional case see [2, 3, 6]. On the other hand when $n \geq 3$ the only result we are aware of is the complement of a ball, that is $\Omega = \bar{B}_1^c$ in which case $c_\Omega = \frac{1}{4}$, see [7, 10].

More generally, for $s > 1$, one can consider weighted Hardy inequalities of the form

$$\int_\Omega \frac{|\nabla u(x)|^2}{d^{s-2}(x)}\,dx \geq c_s(\Omega) \int_\Omega \frac{u^2(x)}{d^s(x)}\,dx, \qquad \forall u \in C_c^\infty(\Omega).$$

By practically the same proof as in the unweighted case, one can show that if Ω is weakly mean convex, then $c_s(\Omega) = \left(\frac{s-1}{2}\right)^2$, under very mild regularity assumptions on the boundary of Ω as before.

Let us now consider the case where Ω is such that Ω^c is bounded with nonempty interior. Then by testing a function that behaves like $d^{\frac{s-1}{2}+\varepsilon}(x)$ near the boundary of Ω and passing to the limit $\varepsilon \to 0^+$, one can easily conclude that $c_s(\Omega) \leq \left(\frac{s-1}{2}\right)^2$. On the other hand by testing a function behaving near infinity like $d^{-\frac{n-s}{2}-\varepsilon}(x)$ and passing to the limit $\varepsilon \to 0^+$ one has that $c_s(\Omega) \leq \left(\frac{n-s}{2}\right)^2$. Therefore we always have

$$c_s(\Omega) \leq \min\left(\left(\frac{s-1}{2}\right)^2, \left(\frac{n-s}{2}\right)^2\right).$$

For $s = 2$ and Ω bounded with smooth boundary, then $c_2(\Omega) \leq \frac{1}{4}$ and the following dichotomy is known. If $c_2(\Omega) = \frac{1}{4}$ then there is no minimizer, whereas when $c_2(\Omega) < \frac{1}{4}$ we have existence of a minimizer, see [10, 4]. If on the other hand Ω^c is bounded with smooth boundary and nonempty interior, the dichotomy now is: when $c_2(\Omega) = \min\left(\frac{1}{4}, \left(\frac{n-2}{2}\right)^2\right)$ there is no minimizer, whereas when $c_2(\Omega) < \min\left(\frac{1}{4}, \left(\frac{n-2}{2}\right)^2\right)$ we have existence of a minimizer, see [5, 9, 10].

One expects that the best constant depends on the geometry of Ω. On the other hand in [1], for $s > n$ and any Ω which is a proper subset of $\mathbb{R}^n$, the following inequality was established

$$\int_\Omega \frac{|\nabla u(x)|^2}{d^{s-2}(x)}\,dx \geq \left(\frac{n-s}{2}\right)^2 \int_\Omega \frac{u^2(x)}{d^s(x)}\,dx, \qquad \forall u \in C_c^\infty(\Omega).$$

In case Ω^c is bounded, one may use the test functions near infinity, mentioned before, and conclude that in fact the constant $\left(\frac{n-s}{2}\right)^2$ is the best one in the case $s > n$. We note that $\left(\frac{n-s}{2}\right)^2 < \left(\frac{s-1}{2}\right)^2$ if and only if $s > \frac{n+1}{2}$.

In this work we initially consider the case $\Omega = \bar{B}_1^c = \{x \in \mathbb{R}^n : \ |x| > 1\}$, whence $d(x) = |x| - 1$. Our first result reads

Theorem 1.1. *Let $n \geq 2$ and $s > 1$. The best constant of the Hardy inequality,*

$$\int_{\bar{B}_1^c} \frac{|\nabla u|^2}{(|x|-1)^{s-2}}dx \geq c(n,s) \int_{\bar{B}_1^c} \frac{u^2}{(|x|-1)^s}dx, \qquad \forall u \in C_c^\infty(\bar{B}_1^c),$$

(i) *in the case $n = 2, 3$, and $1 < s < n$, is given by*

$$c(n,s) = \begin{cases} \left(\frac{s-1}{2}\right)^2, & \text{if} \quad 1 < s \leq \frac{n+1}{2}, \\ \left(\frac{n-s}{2}\right)^2, & \text{if} \quad \frac{n+1}{2} < s < n. \end{cases},$$

and is not realized in the proper energy space,
(ii) *in the case $n > 3$ and $1 < s < n$, is given by*

$$c(n,s) = \begin{cases} \left(\frac{s-1}{2}\right)^2, & \text{if} \quad 1 < s \leq \frac{3n-5}{n-1}, \\ \frac{(n-2)(n-s-1)(s-2)}{(n-3)^2} & \text{if} \quad \frac{3n-5}{n-1} < s < \frac{n^2-3n+4}{n-1} \\ \left(\frac{n-s}{2}\right)^2, & \text{if} \quad \frac{n^2-3n+4}{n-1} \leq s < n. \end{cases}$$

Moreover, when

$$n > 3 \qquad \text{and} \qquad \frac{3n-5}{n-1} < s < \frac{n^2-3n+4}{n-1},$$

the best constant is realized by the function

$$u(x) = |x|^{2-n}(|x|-1)^{\frac{(n-2)(s-2)}{n-3}}, \qquad |x| > 1,$$

whereas in the other cases it is not realized in the proper energy space,
(iii) *in the case $s > n$, is given by*

$$c(n,s) = \left(\frac{n-s}{2}\right)^2,$$

and is not realized in the proper energy space.

Next, for $n \geq 2$, $m \geq 1$ we consider the complement of a cylinder

$$\Omega = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m, \quad |x| > 1, \quad y \in \mathbb{R}^m\} = \bar{B}_1^c \times \mathbb{R}^m \ .$$

Theorem 1.2. *For $n \geq 2$, $m \geq 1$ and $1 < s \neq n$ the following Hardy inequality holds true*

$$\int_{\mathbb{R}^m} \int_{\bar{B}_1^c} \frac{|\nabla_{(x,y)} u(x,y)|^2}{(|x|-1)^{s-2}} \, dx\, dy \geq c(n,s) \int_{\mathbb{R}^m} \int_{\bar{B}_1^c} \frac{u^2(x,y)}{(|x|-1)^s} \, dx\, dy, \qquad \forall u \in C_c^\infty(\bar{B}_1^c \times \mathbb{R}^m),$$

where the constant $c(n,s)$ is the one given by Theorem 1.1 and it is sharp. This time however, the best constant is never realized in the proper energy space.

To find the best constant $c(n,s)$ of Theorem 1.1, we study the existence and the behavior of positive radial solutions of the Euler Lagrange

$$\left(\frac{r^{n-1}\phi'(r)}{(r-1)^{s-2}}\right)' + c(n,s)\frac{r^{n-1}}{(r-1)^s}\,\phi(r) = 0, \qquad r > 1.$$

We make various choices of $c(n,s)$ and in each case, with an appropriate change of variables, we reduce the problem to the study of existence of connecting orbits to a family of singular first order ODEs. These ODEs have a surprisingly rich behaviour depending on the values of n and s and Section 2 is devoted to their detailed study. Finally, in Section 3 we give the proofs of our Theorems.

2 Phase portrait analysis

In this section we will study various singular ODEs that are connected to our problem. Our aim is to establish the existence of connecting orbits between two critical points. The choice of the ODE depends on the parameter s.

2.1 Case 1

Here we consider the case $1 < s < n$ and we will study solutions of the singular ODE,

$$\frac{dy}{dx} = \frac{-(n-1)xy + \frac{s-1}{2}[1 + \frac{2(n-s)}{s-1}y + y^2]}{x(1-x)}, \qquad 0 < x < 1. \tag{2.1}$$

For $s \neq \frac{n+1}{2}$ we denote by

$$\rho_2 < -\frac{n-s}{s-1} < \rho_1 < 0,$$

the roots of

$$H(t) := 1 + 2\frac{n-s}{s-1}t + t^2.$$

There are three critical points of the ODE, namely, $(1,1)$, $(0,\rho_1)$, $(0,\rho_2)$ that will be important to our analysis. There are other critical points, that is, points at which the numerator of the right hand side is zero

$$-(n-1)xy + \frac{s-1}{2}[1 + \frac{2(n-s)}{s-1}y + y^2] = 0, \qquad 0 < x < 1.$$

Clearly, they lie on the curve

$$x = \frac{s-1}{2(n-1)} \frac{1 + \frac{2(n-s)}{s-1}y + y^2}{y}, \qquad 0 < x < 1,$$

which equivalently can be written as

$$x = \frac{s-1}{2(n-1)} \frac{1 + \frac{2(n-s)}{s-1}y + y^2}{y}, \qquad \rho_2 < y < \rho_1.$$

If there is a pair (x_0, y_0) with $x_0 \in (0,1)$ and $y_0 \in (\rho_2, \rho_1)$ such that

$$-(n-1)x_0y_0 + \frac{s-1}{2}\left(1 + \frac{2(n-s)}{s-1}y_0 + y_0^2\right) = 0,$$

then the solution of the ODE with $y(x_0) = y_0$ is such that for all $x \in (0, x_0)$ there holds

$$-(n-1)xy(x) + \frac{s-1}{2}\left(1 + \frac{2(n-s)}{s-1}y(x) + y^2(x)\right) < 0,$$

and therefore y is decreasing and $\lim_{x\to 0^+} y(x) = \rho_1$. On the other hand for $x > x_0$, $y(x)$ is increasing for as long as it exists.

Our interest is to find conditions on the parameters, so that there exists an orbit connecting $(1,1)$ to either $(0,\rho_1)$ or $(0,\rho_2)$. To this end we first have

Lemma 2.1. *Let $n \geq 2$, $s > 1$. (a) There exists an analytic solution $y(x)$ of (2.1) near $(x, y) = (1, 1)$. Moreover for some $\varepsilon > 0$ and any $x \in (1 - \varepsilon, 1]$ there holds*

$$y_a(x) = 1 + (n-1)(x-1) + \frac{n-1}{2}\left(n - 2 - \frac{(s-1)(n-1)}{2}\right)(x-1)^2 + O((x-1)^3).$$

(b) If for some $\varepsilon \in (0,1)$ there exists a solution $y(x)$ of (2.1) in $(1-\varepsilon, 1)$ that in addition satisfies

$$y(x) \geq y_a(x) \quad for \quad x \in (1-\varepsilon, 1) \quad and \quad lim_{x \to 1^-} y(x) = 1,$$

then necessarily $y(x) = y_a(x)$.

Proof: We write the ODE in the following way

$$(x-1)y'(x) = \frac{(n-1)xy - \frac{s-1}{2}\left[1 + \frac{2(n-s)}{s-1}y + y^2\right]}{x} = f(x, y).$$

We next apply Proposition 1.1.1, p. 261 of [8], in a neighbourhood of the point $(x = 1, y = 1)$ since $f(1,1) = 0$ and

$$\frac{\partial f}{\partial y}(1, 1) = 0.$$

The asymptotics at the point $(1, 1)$ follow easily.
(b) Suppose on the contrary there are two solutions $y(x) > y_a(x)$ in $(1 - \varepsilon, 1)$ which tend to 1 as $x \to 1^-$. We define $\phi(x) = y(x) - y_a(x) > 0$. Clearly $\lim_{x \to 1^-} \phi(x) = 0$ and is easily seen that ϕ satisfies the ODE,

$$\phi'(x) = \frac{-(n-1)x\phi(x) + \frac{s-1}{2}\left(\frac{2(n-s)}{s-1} + y(x) + y_a(x)\right)\phi(x)}{x(1-x)}, \qquad 1 - \varepsilon < x < 1.$$

From this we easily derive

$$\frac{\phi(x)}{\phi(1-\varepsilon)} = e^{\int_{1-\varepsilon}^{x} \frac{-(n-1)t + \frac{s-1}{2}\left(\frac{2(n-s)}{s-1} + y(t) + y_a(t)\right)}{t(1-t)}\, dt}.$$

Taking the limit $x \to 1^-$ we arrive at a contradiction: the left hand side tends to zero whereas the right hand side is bounded below by a positive constant since

$$\begin{aligned}\int_{1-\varepsilon}^{x} & \frac{-(n-1)t + \frac{s-1}{2}\left(\frac{2(n-s)}{s-1} + y(t) + y_a(t)\right)}{t(1-t)}\, dt \\ & \geq \int_{1-\varepsilon}^{x} \frac{-(n-1)t + (s-1)\left(\frac{n-s}{s-1} + y_a(t)\right)}{t(1-t)}\, dt,\end{aligned}$$

and the right hand side is finite because of the asymptotics of y_a. This completes the proof of part (b).

$\square$

Lemma 2.2. *Let $n \geq 2$ and $1 < s < \frac{n+1}{2}$. Then* *(a) there exists an analytic solution $y_0(x)$ near zero that solves ODE (2.1) and such that for some $\varepsilon > 0$,*

$$y_0(x) = \rho_2 + \frac{(n-1)\rho_2}{n-s-1+(s-1)\rho_2}x + O(x^2), \qquad x \in [0,\varepsilon).$$

(b) If for some $\varepsilon \in (0,1)$, there exists solution $y(x)$ of the ODE (2.1) in $(0,\varepsilon)$ with the property $\lim_{x\to 0^+} y(x) = \rho_2$, then necessarily

$$y(x) = y_0(x), \qquad x \in [0,\varepsilon).$$

Proof: (a) We write the ODE as

$$xy'(x) = \frac{-(n-1)xy + \frac{s-1}{2}[1 + \frac{2(n-s)}{s-1}y + y^2]}{1-x} = f(x,y).$$

We next apply Proposition 1.1.1, p. 261 of [8], in a neighbourhood of the point $(x = 0, y = \rho_2)$ since $f(0,\rho_2) = 0$ and

$$\frac{\partial f}{\partial y}(0,\rho_2) = (n-s) + (s-1)\rho_2 < 0.$$

The asymptotics at the point $(0,\rho_2)$ follow easily.
(b) Suppose on the contrary there are two solutions $y_1(x) > y_2(x)$ in $(0,\varepsilon)$ which tend to ρ_2 as $x \to 0^+$. We define $\phi(x) = y_1(x) - y_2(x)$. Clearly $\lim_{x\to 0^+}\phi(x) = 0$ and is easily seen that ϕ satisfies the ODE,

$$\phi'(x) = \frac{-(n-1)x\phi(x) + \frac{s-1}{2}\left(\frac{2(n-s)}{s-1} + y_1(x) + y_2(x)\right)\phi(x)}{x(1-x)}, \qquad 0 < x < \varepsilon.$$

From this we easily derive

$$\frac{\phi(x)}{(1-x)^{n-1}} = \frac{\phi(\varepsilon)}{(1-\varepsilon)^{n-1}} e^{\frac{s-1}{2}\int_\varepsilon^x \frac{\frac{2(n-s)}{s-1} + y_1(t) + y_2(t)}{t(1-t)}dt}.$$

Taking the limit $x \to 0^+$ we arrive at a contradiction: the left hand side tends to zero whereas the right hand side tends to infinity since

$$\lim_{t\to 0^+}\left(\frac{2(n-s)}{s-1} + y_1(t) + y_2(t)\right) = 2\left(\frac{n-s}{s-1} + \rho_2\right) < 0.$$

The result then follows from part (a).

$\square$

Lemma 2.3. *Let $n \geq 2$. Then*

$$\overline{y}(x) = \rho_2 + (1-\rho_2)x, \qquad 0 < x < 1,$$

is a supersolution of the ODE (2.1) provided that
- *either $2 \leq n \leq 3$ and $1 < s \leq \frac{n+1}{2}$*
- *or $n > 3$ and $1 < s \leq \frac{3n-5}{n-1}$.*

Proof: For $\overline{y}$ to be a supersolution we should have, for $0 < x < 1$,

$$\begin{aligned}(1-\rho_2) \quad \geq \quad & \frac{-(n-1)x(\rho_2+(1-\rho_2)x)}{x(1-x)} \\ & + \frac{\frac{s-1}{2}\left(1+\frac{2(n-s)}{s-1}(\rho_2+(1-\rho_2)x)+(\rho_2+(1-\rho_2)x)^2\right)}{x(1-x)}.\end{aligned}$$

After straightforward calculations this is equivalent to

$$(s-1)\rho_2^2+(2n-2s-1)\rho_2-(n-s-1)-(1-\rho_2)\left(2-n+\frac{s-1}{2}(1-\rho_2)\right)x \geq 0$$

for $0 < x < 1$. This is true provided that

$$(s-1)\rho_2^2+(2n-2s-1)\rho_2-(n-s-1) \geq 0 \tag{2.2}$$

and

$$(s-1)\rho_2^2+(2n-2s-1)\rho_2-(n-s-1)+(1-\rho_2)\left(2-n+\frac{s-1}{2}(1-\rho_2)\right) \geq 0. \tag{2.3}$$

Using the fact that ρ_2 satisfies

$$1+\frac{2(n-s)}{s-1}\rho_2+\rho_2^2=0,$$

inequality (2.2) is equivalent to

$$\rho_2 \leq -(n-2), \tag{2.4}$$

whereas in (2.3) the left hand side is exactly equal to zero. Hence it remains to establish (2.4). From (2.2) we get that

$$\rho_2 = -\frac{n-s+\sqrt{(n+1-2s)(n-1)}}{s-1},$$

so that (2.4) is equivalent to

$$\sqrt{(n+1-2s)(n-1)} \geq (s-2)(n-1). \tag{2.5}$$

When $2 \leq n \leq 3$ and $s < 2$ we have strict inequality whereas for $n = 3$ and $s = 2$ we have equality and in addition $\rho_2 = -1$ and $y(x) = -1 + 2x$ is a solution.

For $n > 3$ and $s \leq 2$, strict inequality (2.5) is obvious, whereas for $s > 2$ inequality (2.5) is equivalent to

$$(s-1)\left(s-\frac{3n-5}{n-1}\right) \leq 0,$$

whence the result. We note that $\rho_2 = -(n-2)$ iff $s = \frac{3n-5}{n-1}$ in which case

$$y(x) = -(n-2)+(n-1)x, \qquad 0 < x < 1,$$

is a solution of the ODE.

$\square$

Theorem 2.4. *(a) If $2 \leq n < 3$ and $1 < s \leq \frac{n+1}{2}$ or $n = 3$ and $1 < s < 2$ then there is a solution $y_a(x)$ analytic at $(x, y) = (1, 1)$ that is defined for all $x \in (0, 1)$ and in addition $\lim_{x\to 0^+} y_a(x) = \rho_1$.*

(b) If $n > 3$ and $1 < s < \frac{3n-5}{n-1}$ then then there is a solution $y_a(x)$ analytic at $(x, y) = (1, 1)$ that is defined for all $x \in (0, 1)$ and in addition $\lim_{x\to 0^+} y_a(x) = \rho_1$.

(c) If $n \geq 3$ and $s = \frac{3n-5}{n-1}$ then $y_a(x) = -(n-2) + (n-1)x$ is the analytic solution. connecting $(0, -(n-2))$ with $(1, 1)$.
Furthermore,

(d) under the hypothesis of either (a) or (b), the analytic solution y_0 of Lemma 2.2 is defined for all $x \in (0, 1)$ and connects $(0, \rho_2)$ with $(1, 1)$. In addition, there is a continuum of solutions connecting $(0, \rho_2)$ with $(1, 1)$. These solutions lie between y_0 and y_a and these are the only bounded solutions of (2.1) in $(0, 1)$.

(e) under the hypothesis of (c), $y_a(x) = -(n-2)+(n-1)x$ is the only bounded solution of the ODE.

Proof: An easy computation shows that under any of our assumptions

$$\rho_2 < -(n-2) \quad \Leftrightarrow \quad \frac{(n-1)\rho_2}{n-s-1+(s-1)\rho_2} < 1 - \rho_2. \tag{2.6}$$

From Lemma 2.3 we have $\rho_2 < -(n-2)$ whenever $2 \leq n \leq 3$ and $1 < s \leq \frac{n+1}{2}$ with the exception of $n = 3$, $s = 2$ where equality holds. We also have $\rho_2 < -(n-2)$ whenever $n > 3$ and $1 < s < \frac{3n-5}{n-1}$. Whenever $n > 3$ and $s = \frac{3n-5}{n-1}$ equality holds $\rho_2 = -(n-2)$ in which case

$$y_a(x) = -(n-2) + (n-1)x, \qquad 0 < x < 1,$$

is a solution of the ODE.

In the sequel we consider the case $\rho_2 < -(n-2)$. From the asymptotics of the analytic at $(1, 1)$ solution we initially have $y_a(x) > \rho_2 + (1-\rho_2)x = \overline{y}(x)$ for x close to 1. Using Lemma 2.3 and comparison we conclude that

$$y_a(x) > \rho_2 + (1-\rho_2)x, \qquad \forall x \in (0, 1). \tag{2.7}$$

We next consider the following two cases:
(i) If for all $x \in (0, 1)$,

$$-(n-1)xy_a(x)+\frac{s-1}{2}[1+\frac{2(n-s)}{s-1}y_a(x)+y_a^2(x)] = -(n-1)xy_a(x)+\frac{s-1}{2}H(y_a(x)) > 0,$$

then the solution is monotonic and has a finite limit which is either ρ_1 or ρ_2.
(ii) If there is a point $x_0 \in (0, 1)$ such that

$$-(n-1)xy_a(x_0) + \frac{s-1}{2}H(y_a(x_0)) = 0,$$

then from the ODE it follows that for all $x \in (0, x_0)$ there holds

$$-(n-1)xy_a(x) + \frac{s-1}{2}H(y_a(x)) < 0,$$

and therefore $y_a(x)$ monotonically tends to ρ_1 as $x \to 0^+$.

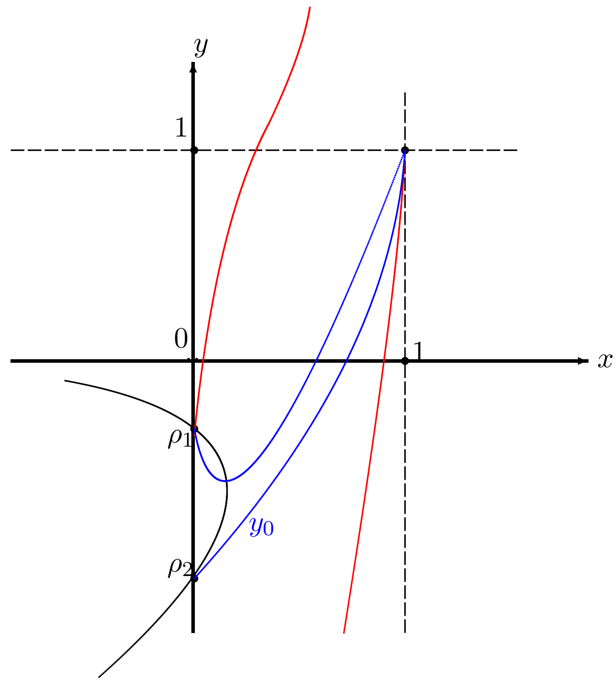

Figure 1: Either $n = 2, 3$ and $1 < s < \frac{n+1}{2}$ or else $n > 3$ and $1 < s < \frac{3n-5}{n-1}$. Connecting orbits are in blue. Blowing up ones in red.

We next return to case (i) and exclude the case $\lim_{x\to 0^+} y_a(x) = \rho_2$. Suppose that it tends to ρ_2. Using the uniqueness of Lemma 2.2, $y_a(x)$ should be analytic at $(0, \rho_2)$. But then, by passing to the limit in (2.7) we would have that

$$\frac{(n-1)\rho_2}{n-s-1+(s-1)\rho_2} = y_a'(0) \geq 1 - \rho_2,$$

which contradicts (2.6). This concludes the proof for the cases (a) and (b); see fig. 1. Case (c) is obvious.

(d) The analytic solution y_0 of Lemma 2.2 stays below y_a and consequently it tends to $(1, 1)$. Let $x_0 \in (0, 1)$ and $y_0 \in (y_0(x_0), y_a(x_0))$ then the solution of (2.1) with $y(x_0) = y_0$ satisfies $y_0(x) < y(x) < y_a(x)$ and $\lim_{x\to 1^-} y(x) = 1$. By a similar argument as in case (i) or (ii) above we also have $\lim_{x\to 0^+} y(x) = \rho_1$. If on the other hand $y(x_0) < y_0(x_0)$ for some $x_0 \in (0, 1)$, then $\lim_{x\to 1^-} y(x) = 1$ and y blows up to the left at some $x_1 \in (0, x_0)$ due to Lemma 2.2(b). Similarly if for some $x_0 \in (0, 1)$, $y(x_0) > y_a(x_0)$, then this solution tends to to ρ_1 as $x \to 0^+$ and blows up at some $x_2 \in (x_0, 1)$, due to Lemma 2.1(b).

(e) If for some $x_0 \in (0, 1)$, $y(x_0) \neq y_a(x_0)$ then the solution $y(x)$ will blow up in a similar fashion as in case (d).

$\square$

Remark When either $n = 2, 3$ and $\frac{n+1}{2} < s < n$ or else $n > 3$ and $\frac{3n-5}{n-1} < s < n$, one can establish that there are no connecting orbits and all solutions blow up; see fig.2.

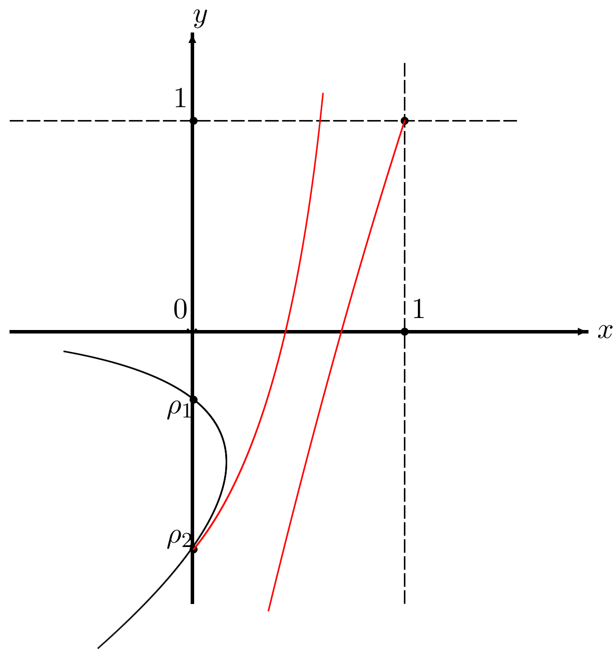

Figure 2: Either $n = 2, 3$ and $\frac{n+1}{2} < s < n$ or else $n > 3$ and $\frac{3n-5}{n-1} < s < n$. There are no connecting orbits. Solutions in red blow up.

2.2 Case 2

Here we will consider the case $n > 3$ and $2 < s < n-1$. We will study solutions of the singular ODE

$$\frac{dy}{dx} = \frac{-(n-1)xy + \frac{(n-2)(s-2)}{n-3}\left[\frac{n-s-1}{(n-2)(s-2)} + \frac{(n-3)(n-s)}{(n-2)(s-2)}y + y^2\right]}{x(1-x)} . \tag{2.8}$$

The roots of

$$\frac{n-s-1}{(n-2)(s-2)} + \frac{(n-3)(n-s)}{(n-2)(s-2)}t + t^2$$

are $-\frac{n-s-1}{s-2}$ and $-\frac{1}{n-2}$. We note that

$$-\frac{n-s-1}{s-2} < -\frac{1}{n-2} \quad \Leftrightarrow \quad s < \frac{n^2-3n+4}{n-1} .$$

At $x = 1$ the roots of

$$-(n-1)t + \frac{(n-2)(s-2)}{n-3}\left[\frac{n-s-1}{(n-2)(s-2)} + \frac{(n-3)(n-s)}{(n-2)(s-2)}t + t^2\right] = 0,$$

or equivalently,

$$\frac{n-s-1}{(n-2)(s-2)} - \frac{(n-3)(s-1)}{(n-2)(s-2)}t + t^2 = 0.$$

are 1 and $\frac{n-s-1}{(n-2)(s-2)}$. We note that

$$1 < \frac{n-s-1}{(n-2)(s-2)} \quad \Leftrightarrow \quad 2 < s < \frac{3n-5}{n-1}.$$

The important critical points of the ODE are

$$(1,1), \quad \left(1, \frac{n-s-1}{(n-2)(s-2)}\right), \quad (0, -\frac{n-s-1}{s-2}), \quad (0, -\frac{1}{n-2}).$$

There are other critical points, that is, points at which the numerator of the right hand side is zero

$$-(n-1)xy+\frac{(n-2)(s-2)}{n-3}\left[\frac{n-s-1}{(n-2)(s-2)}+\frac{(n-3)(n-s)}{(n-2)(s-2)}y+y^2\right]=0 \qquad 0<x<1.$$

Clearly, they lie on the curve

$$x=\frac{(n-2)(s-2)}{(n-3)(n-1)}\left[\frac{\frac{n-s-1}{(n-2)(s-2)}+\frac{(n-3)(n-s)}{(n-2)(s-2)}y+y^2}{y}\right]=:P_2(y), \qquad 0<x<1.$$

Now there are two branches corresponding to $y>0$ and $y<0$.

If there is a pair (x_0,y_0) on the curve with $x_0\in(0,1)$ and $y_0<0$, then the solution of the ODE (2.8) is such that for all $x\in(0,x_0)$ there holds

$$-(n-1)xy(x)+\frac{(n-2)(s-2)}{n-3}\left[\frac{n-s-1}{(n-2)(s-2)}+\frac{(n-3)(n-s)}{(n-2)(s-2)}y(x)+y^2(x)\right]<0\ . \tag{2.9}$$

Consequently $y(x)$ is decreasing in $(0,x_0)$.

Similarly, if there is a pair (x_0,y_0) on the curve with $x_0\in(0,1)$ and $y_0>0$, then the solution of the ODE (2.8) is such that for all $x\in(x_0,1)$ inequality (2.9) holds and therefore $y(x)$ is decreasing in $(x_0,1)$. Outside these regions the solution is increasing.

Lemma 2.5. *Let $n>3$ and $2<s<n-1$. Then*

$$y_a(x)=-\frac{n-s-1}{s-2}+\frac{n-3}{s-2}x, \qquad 0<x<1,$$

is an analytic solution of the ODE (2.8) connecting $(1,1)$ to $(0,-\frac{n-s-1}{s-2})$, whereas

$$\underline{y}(x)=-\frac{1}{n-2}+\frac{n-3}{(n-2)(s-2)}x, \qquad 0<x<1,$$

is a subsolution.

Proof: Both statements follow by straightforward calculations.

$\square$

Lemma 2.6. *Let $n>3$. (a) If $\frac{3n-5}{n-1}<s<n-1$ and for some $\varepsilon\in(0,1)$ there exists solution $y(x)$ of the ODE (2.8) in $(1-\varepsilon,1)$ with the property $\lim_{x\to1^-}y(x)=1$, then necessarily*

$$y(x)=y_a(x), \qquad x\in(1-\varepsilon,1).$$

(b) If $2<s<\frac{n^2-3n+4}{n-1}$ and for some $\varepsilon\in(0,1)$ there exists solution $y(x)$ of the ODE (2.8) in $(0,\varepsilon)$ with the property $\lim_{x\to0^+}y(x)=-\frac{n-s-1}{s-2}$ then necessarily

$$y(x)=y_a(x), \qquad x\in(0,\varepsilon).$$

Proof: (a) Suppose on the contrary there are two solutions $y_1(x)>y_2(x)$ in $(1-\varepsilon,1$ which tend to 1 as $x\to1^-$. We define $\phi(x)=y_1(x)-y_2(x)$. Clearly $\lim_{x\to1^-}\phi(x)=0$ and is easily seen that ϕ satisfies the ODE, for $1-\varepsilon<x<1$,

$$\phi'(x)=\frac{-(n-1)x\phi(x)+\frac{(n-2)(s-2)}{n-3}\left(\frac{(n-3)(n-s)}{(n-2)(s-2)}+y_1(x)+y_2(x)\right)\phi(x)}{x(1-x)}.$$

From this we easily derive for $1-\varepsilon < x < 1$,

$$\phi(x) = \phi(1-\varepsilon)\, e^{\int_{1-\varepsilon}^{x} \frac{-(n-1)t + \frac{(n-2)(s-2)}{n-3}\left(\frac{(n-3)(n-s)}{(n-2)(s-2)} + y_1(t) + y_2(t)\right)}{t(1-t)}\, dt}.$$

Taking the limit $x \to 1^-$ we arrive at a contradiction: the left hand side tends to zero whereas the right hand side tends to infinity since

$$\begin{aligned}
&\lim_{t\to 1^-}\left[-(n-1)t + \frac{(n-2)(s-2)}{n-3}\left(\frac{(n-3)(n-s)}{(n-2)(s-2)} + y_1(t) + y_2(t)\right)\right] \\
&\quad = \frac{(n-1)s - 3n + 5}{n-3} > 0.
\end{aligned}$$

The proof of part (b) is quite similar. In particular it follows from the fact that

$$\begin{aligned}
&\lim_{t\to 0^+}\left[-(n-1)t + \frac{(n-2)(s-2)}{n-3}\left(\frac{(n-3)(n-s)}{(n-2)(s-2)} + y_1(t) + y_2(t)\right)\right] \\
&\quad = \frac{(n-1)s - (n^2-3n+4)}{n-3} < 0.
\end{aligned}$$

□

We next state two lemmas

Lemma 2.7. *Let $n > 3$ and $2 < s < \frac{3n-5}{n-1}$. Then*
(a) there exists an analytic solution $y_1(x)$ near $x = 1$ that solves ODE (2.8) and such that for some $\varepsilon > 0$ and $x \in [1-\varepsilon, 1)$

$$y_1(x) = \frac{n-s-1}{(n-2)(s-2)} + \frac{(n-1)(n-3)(n-s-1)}{(n-2)(s-2)(4(n-2)-(n-1)s)}(x-1) + O((x-1)^2).$$

(b) If for some $\varepsilon \in (0,1)$, there exists solution $y(x)$ of the ODE (2.8) in $(1-\varepsilon, 1)$ with the property $\lim_{x\to 1^-} y(x) = \frac{n-s-1}{(n-2)(s-2)}$, then necessarily

$$y(x) = y_1(x), \qquad x \in (1-\varepsilon, 1).$$

Lemma 2.8. *Let $n > 3$ and $\frac{n^2-3n+4}{n-1} < s < n-1$ Then*
(a) there exists an analytic solution $y_0(x)$ near $x = 0$ that solves ODE (2.8) and such that for some $\varepsilon > 0$ and $x \in (0, \varepsilon)$

$$y_0(x) = -\frac{1}{n-2} + \frac{(n-1)(n-3)}{(n-2)((n-1)s - (n^2-3n+4) + n-3)}\, x + O(x^2).$$

(b) If for some $\varepsilon \in (0,1)$, there exists solution $y(x)$ of the ODE (2.8) in $(\varepsilon, 0)$ with the property $\lim_{x\to 0^+} y(x) = -\frac{1}{n-2}$, then necessarily

$$y(x) = y_0(x), \qquad x \in (0, \varepsilon).$$

The proof of the above two Lemmas is quite similar to the proof of Lemma 2.2 or 2.6 and we omit them.

Theorem 2.9. *Let $n > 3$ and $2 < s < n-1$. We recall that*

$$y_a(x) = -\frac{n-s-1}{s-2} + \frac{n-3}{s-2}x.$$

is an analytic solution of the ODE connecting $(0, -\frac{n-s-1}{s-2})$ to $(1,1)$. In addition to y_a,
(a) in case $2 < s < \frac{3n-5}{n-1}$ there is a continuum of orbits connecting $(0, -\frac{1}{n-2})$ with $(1,1)$ and these are all the bounded solutions in $(0,1)$,
(b) in case $\frac{3n-5}{n-1} < s < \frac{n^2-3n+4}{n-1}$ there is no other bounded solution in $(0,1)$,
(c) in case $\frac{n^2-3n+4}{n-1} < s < n-1$ there is a continuum of orbits connecting $(0, -\frac{n-s-1}{s-2})$ to $(1, \frac{n-s-1}{(n-2)(s-2)})$ and these are all the bounded solutions in $(0,1)$.

Proof: We initially observe that the line $\underline{y}(x) = -\frac{1}{n-2} + \frac{n-3}{(n-2)(s-2)}x$ and the curve $x = P_2(y)$ have only two points of intersection, namely, $(0, -\frac{1}{n-2})$ and $\left(1, \frac{n-s-1}{(n-2)(s-2)}\right)$.

Similarly the line $y_a(x) = -\frac{n-s-1}{s-2} + \frac{n-3}{s-2}x$ and the curve $x = P_2(y)$ intersect each other at the points $(0, -\frac{n-s-1}{s-2})$ and $(1,1)$.

(a) Let (x_0, y_0) a point on the curve $x = P_2(y)$ with $1 < y_0 < \frac{n-s-1}{(n-2)(s-2)}$. The solution of the ODE with $y(x_0) = y_0$ is such that for $x \in (x_0, 1)$, $y(x)$ decreases to 1. By comparison $y_a(x) < y(x) < \underline{y}(x)$, and because there is only one solution which tends to $-\frac{n-s-1}{s-2}$ as $x \to 0^+$, cf Lemma 2.6(b), we conclude that $\lim_{x\to 0^+} y(x) = -\frac{1}{n-2}$. By a similar argument the analytic solution $y_1(x)$ of Lemma 2.7 tends as $x \to 0^+$ to $-\frac{1}{n-2}$.

Any other solution of (2.8) $y(x)$, which at some point $x_0 \in (0,1)$ is below the analytic, that is,

$$y_0 < y_a(x_0) = -\frac{n-s-1}{s-2} + \frac{n-3}{s-2}x_0, \qquad y(x_0) = y_0,$$

to the right connects to (1,1) and to the left blows up at some point $x_* \in (0,1)$. The last statement follows from Lemma 2.6(b). A similar argument shows that if a solution at a certain point is above y_1 then to the left connects to $(0, -\frac{1}{n-2})$ and to the right blows up; see fig. 3.

(b) In this case using once again Lemma 2.6 we conclude that a solution which is below y_a connects to $\left(1, \frac{n-s-1}{(n-2)(s-2)}\right)$ and blows up to the left and similarly if it is above y_a then it connect to $(0, -\frac{1}{n-2})$ and blows up to the right.

(c) It is easy to check that the analytic solution y_0 of Lemma 2.8, satisfies $y_0(x) > \underline{y}(x)$, $x \in (0,1)$ and connects to $\left(1, \frac{n-s-1}{(n-2)(s-2)}\right)$. Let (x_0, y_0) be on the curve $x = P_2(y)$ with $-\frac{1}{n-2} < y < -\frac{n-s-1}{s-2}$. Then the solution of (2.8) with $y(x_0) = y_0$ connects to the left to $(0, -\frac{1}{n-2})$ and to the right to $\left(1, \frac{n-s-1}{(n-2)(s-2)}\right)$. Any solution below $y_0(x)$ or above $y_a(x)$ blows up in a similar fashion as in part (a).

□

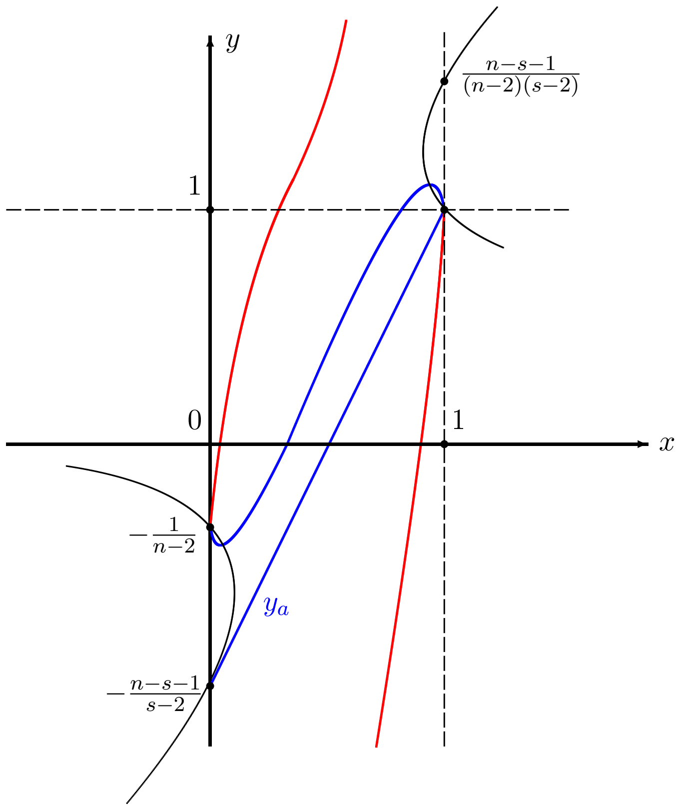

Figure 3: Case $n > 3$ and $2 < s < \frac{3n-5}{n-1}$. Connecting orbits are in blue and blowing up ones in red

2.3 Case 3

Here we will consider the case $n > 3$ and $\frac{n+1}{2} < s < n$ and we will study solutions of the singular ODE

$$\frac{dy}{dx} = \frac{-(n-1)xy + \frac{n-s}{2}(1+y)^2}{x(1-x)}, \qquad 0 < x < 1. \tag{2.10}$$

At $x = 1$ the roots of the equation

$$-(n-1)y + \frac{n-s}{2}(1+y)^2 = 0, \qquad \Leftrightarrow \qquad y^2 - \frac{2(s-1)}{n-s}y + 1 = 0,$$

satisfy

$$0 < \tau_2 < \frac{s-1}{n-s} < 1 < \tau_1.$$

The important critical points of the ODE are

$$(0, -1), \qquad (1, \tau_1), \qquad (1, \tau_2)\ .$$

There are other critical points that lie on the curve

$$-(n-1)xy + \frac{n-s}{2}(1+y)^2 = 0, \qquad 0 < x < 1.$$

If there is a pair (x_0, y_0) on the curve with $x_0 \in (0,1)$ then the solution of the ODE with $y(x_0) = y_0$ is such that for $x \in (x_0, 1)$ satisfies

$$-(n-1)xy(x) + \frac{n-s}{2}(1+y(x))^2 = 0, \qquad x_0 < x < 1,$$

and it decreases to τ_2. For $x < x_0$ we have the opposite sign and the solutions are increasing as long as they exist.

Lemma 2.10. *Let $n \geq 2$, $1 < s < n$. (a) There exists an analytic solution $y(x)$ of (2.10) near $(x,y) = (0,-1)$. Moreover for some $\varepsilon > 0$ and any $x \in [0,\varepsilon)$ there holds*

$$y_a(x) = -1 + (n-1)x + \frac{n-1}{2}\left(-(n-2) + \frac{(n-s)(n-1)}{2}\right)x^2 + O(x^3).$$

(b) If for some $\varepsilon \in (0,1)$ there exists a solution $y(x)$ of (2.10) in $(0,\varepsilon)$ that in addition satisfies

$$y(x) \leq y_a(x) \ \ for \ \ x \in (0,\varepsilon) \ \ and \ \ lim_{x\to 0^+} y(x) = -1,$$

then necessarily $y(x) = y_a(x)$.

Proof: (a) We write the ODE in the following way

$$xy'(x) = \frac{-(n-1)xy + \frac{n-s}{2}(1+y)^2}{(1-x)} = f(x,y).$$

We next apply Proposition 1.1.1. p. 261 of [8], in a neighbourhood of the point $(x = 0, y = -1)$ since $f(0,-1) = 0$ and

$$\frac{\partial f}{\partial y}(0,-1) = 0.$$

The asymptotics follow easily.
(b) Suppose on the contrary there are two solutions $y_a(x) > y(x)$ in $(0,\varepsilon)$ which tend to -1 as $x \to 0^+$. We define $\phi(x) = y_a(x) - y(x) > 0$. Clearly $\lim_{x\to 0^+} \phi(x) = 0$ and is easily seen that ϕ satisfies the ODE,

$$\phi'(x) = \frac{-(n-1)x\phi(x) + \frac{n-s}{2}\left(2 + y_a(x) + y(x)\right)\phi(x)}{x(1-x)}, \qquad 0 < x < \varepsilon.$$

From this we easily derive

$$\frac{\phi(x)}{\phi(\varepsilon)} = e^{\int_\varepsilon^x \frac{-(n-1)t + \frac{n-s}{2}(2+y_a(t)+y(t))}{t(1-t)}dt}.$$

Taking the limit $x \to 0^+$ we arrive at a contradiction: the left hand side tends to zero whereas the right hand side is bounded below by a positive constant since

$$\int_\varepsilon^x \frac{-(n-1)t + \frac{n-s}{2}\left(2 + y_a(t) + y(t)\right)}{t(1-t)}\,dt \geq \int_\varepsilon^x \frac{-(n-1)t + (n-s)\left(y_a(t) + 1\right)}{t(1-t)}\,dt.$$

This completes the proof of part (b).

$\square$

Lemma 2.11. *Let $n \geq 2$ and $\frac{n+1}{2} < s < n$. Then*
(a) there exists an analytic solution $y_*(x)$ near $(x,y) = (1,\tau_1)$ that solves ODE (2.10) and such that for some $\varepsilon > 0$,

$$y_*(x) = \tau_1 + \frac{(n-1)\tau_1}{(n-s)\tau_1 - (s-2)}(x-1) + O((x-1)^2), \qquad x \in (1-\varepsilon, 1].$$

(b) If for some $\varepsilon \in (0,1)$, there exists solution $y(x)$ of the ODE (2.1) in $(1-\varepsilon, 1)$ with the property $\lim_{x\to 1^-} y(x) = \tau_1$, then necessarily

$$y(x) = y_*(x), \qquad x \in (1-\varepsilon, 1].$$

Proof: (a) We first write the ODE in the following way

$$(x-1)y'(x) = \frac{(n-1)xy - \frac{n-s}{2}(1+y)^2}{x} = f(x,y).$$

We next apply Proposition 1.1.1. p. 261 of [8], in a neighbourhood of the point $(x = 1, y = \tau_1)$ since $f(1,\tau_1) = 0$ and

$$\frac{\partial f}{\partial y}(1,\tau_1) = (s-1) - (n-s)\tau_1 < 0.$$

The asymptotics at the point $(1,\tau_1)$ follow easily.

(b) Suppose on the contrary there are two such solutions $y_1(x) > y_2(x)$ in $(1-\varepsilon, 1)$. We define $\phi(x) = y_1(x) - y_2(x)$. Clearly $\lim_{x\to 1^-} \phi(x) = 0$ and is easily seen that ϕ satisfies the following ODE

$$\phi'(x) = \frac{-(n-1)x\phi(x) + \frac{n-s}{2}(2 + y_1(x) + y_2(x))\phi(x)}{x(1-x)}, \qquad 1-\varepsilon < x < 1.$$

From this we easily derive

$$\phi(x) = \phi(1-\varepsilon)\ e^{\int_{1-\varepsilon}^{x} \frac{-(n-1)t + \frac{n-s}{2}(2+y_1(t)+y_2(t))}{t(1-t)}\,dt}.$$

Taking the limit $x \to 1^-$ we arrive at a contradiction: the left hand side tends to zero and the right hand side tends to infinity since

$$\lim_{t\to 1^-}\left(-(n-1)t + \frac{n-s}{2}(2 + y_1(t) + y_2(t))\right) = -(s-1) + (n-s)\tau_1 > 0.$$

Hence, by part (a) the result follows. $\square$

Lemma 2.12. *Let*
- *either $2 \leq n \leq 3$ and $\frac{n+1}{2} < s < n$*
- *or else $n > 3$ and*

$$\frac{n^2 - 3n + 4}{n-1} < s < n.$$

Then $\tau_1 > n-2$ and in addition

$$\overline{y}(x) = -1 + (1+\tau_1)x, \qquad 0 < x < 1,$$

is a supersolution of the ODE (2.10).

Proof: Inequality $\tau_1 > n-2$ is equivalent to

$$\sqrt{2s-(n+1)} > \sqrt{n-1}\,(n-1-s). \tag{2.11}$$

In case $2 \le n \le 3$, this is clearly true if $n-1 \le s < n$.

Consider now the case $n > 3$. Again, if $n-1 \le s < n$, inequality (2.11) is true. For $\frac{n^2-3n+4}{n-1} < s < n-1$, after squaring, (2.11) is equivalent to

$$\left(s - \frac{n^2-3n+4}{n-1}\right)(s-n) < 0,$$

and the result follows.

For $\overline{y}$ to be a strict supersolution we should have, for $0 < x < 1$,

$$(1+\tau_1) > \frac{-(n-1)x(-1+(1+\tau_1)x) + \frac{n-s}{2}(1+\tau_1)^2x^2}{x(1-x)}.$$

After straightforward calculations this is equivalent to

$$\tau_1 - (n-2) + \left((n-2)(1+\tau_1) - \frac{n-s}{2}(1+\tau_1)^2\right)x > 0, \qquad 0 < x < 1.$$

This is true since at $x = 0$, $\tau_1 > n-2$, and at $x = 1$, the left hand side is identical equal to zero.

□

Theorem 2.13. *Let*
- *either* $2 \le n \le 3$ *and* $\frac{n+1}{2} < s < n$
- *or else* $n > 3$ *and*

$$\frac{n^2-3n+4}{n-1} < s < n.$$

Then (a) there is a solution $y_a(x)$ *of (2.10) which is analytic at* $(x, y) = (0, -1)$, *is defined for all* $x \in (0, 1)$ *and in addition* $\lim_{x\to 1^-} y_a(x) = \tau_2$.
(b) The analytic solution y_* *of Lemma 2.11 is defined for all* $x \in (0, 1)$ *and it connects* $(0, -1)$ *to* $(1, \tau_1)$.
(c) In addition to the above two analytic solutions, there is a continuum of solutions connecting $(0, -1)$ *to* $(1, \tau_2)$, *and these are the only bounded solutions of (2.10) in* $(0, 1)$.

Proof: (a) Actually, y_a is the analytic solution of (2.10) near $(0, -1)$ and we will establish that it is defined for all $x \in (0, 1)$ and it has the required properties.

From Lemma (2.10) the analytic solution near $x = 0$ behaves like

$$y_a(x) = -1 + (n-1)x + O(x^2),$$

therefore for x near zero,

$$y_a(x) < \overline{y}(x) = -1 + (1+\tau_1)x$$

and then using Lemma 2.12 and comparison arguments we deduce

$$y_a(x) < \overline{y}(x) = -1 + (1+\tau_1)x, \qquad 0 < x < 1.$$

As long as $y_a < 0$, y_a is increasing and we next consider the following two cases:
(i) If for all $x \in (0,1)$,

$$-(n-1)xy_a(x) + \frac{n-s}{2}(1+y_a(x))^2 > 0$$

the solution is monotonic and has a finite limit which is either τ_1 or τ_2.
(ii) If there is a point $x_0 \in (0,1)$ such that

$$-(n-1)x_0y_a(x_0) + \frac{n-s}{2}(1+y_a(x_0))^2 = 0,$$

then from the ODE it follows that for all $x \in (x_0,1)$ there holds

$$-(n-1)xy_a(x) + \frac{n-s}{2}(1+y_a(x))^2 < 0,$$

and $y_a(x)$ decreases to τ_2 as $x \to 1^-$.

We next return to case (i) and exclude the case $\lim_{x\to 0^+} y_a(x) = \tau_1$. Suppose that it tends to τ_1. Then by Lemma (2.11) it should be the analytic solution at $(1,\tau_1)$, given by Lemma 2.11, which for x close to one behaves like

$$y_a(x) = \tau_1 + \frac{(n-1)\tau_1}{(n-s)\tau_1 - (s-2)}(x-1) + O((x-1)^2).$$

Assuming that y_a is the analytic solution, since

$$y_a(x) < \overline{y}(x) = \tau_1 + (1+\tau_1)(x-1), \qquad 0 < x < 1.$$

one should have

$$\frac{(n-1)\tau_1}{(n-s)\tau_1 - (s-2)} = y_a'(1) = y'(1) \geq 1 + \tau_1.$$

But this is a contradiction since the opposite inequality holds true. Indeed,

$$\frac{(n-1)\tau_1}{(n-s)\tau_1 - (s-2)}(x-1) < 1 + \tau_1,$$

since this is equivalent to

$$\tau_1 > n-2,$$

which is true by Lemma 2.12.
(b) Using the asymptotics of y_* and comparison it is easy to see that y_* has the required property since $y_*(x) > \overline{y}(x)$.
(c) Any solution that for some $x_0 \in (0,1)$ satisfies $y_a(x_0) < y(x_0) < \overline{y}(x_0)$, stays in between for all $x_0 \in (0,1)$ and connects $(0,-1)$ to $(1,\tau_2)$.

If on the other hand $y(x_0) < y_a(x_0)$ for some $x_0 \in (0,1)$, then $\lim_{x\to 1^-} y(x) = \tau_2$ and blows up to the left at some $x_1 \in (0,x_0)$ due to Lemma 2.10(b). Similarly if for some $x_0 \in (0,1)$, $y(x_0) > y_*(x_0)$, then this solution tends to to -1 as $x \to 0^+$ and blows up at some $x_2 \in (x_0,1)$, due to Lemma 2.11(b); see figure 4.

□

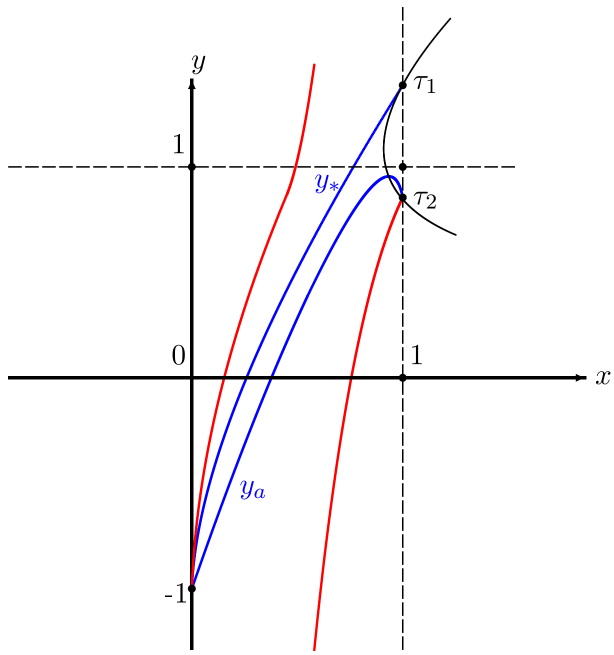

Figure 4: Either $n = 2, 3$ and $\frac{n+1}{2} < s < n$ or else $n > 3$ and $\frac{n^2-3n+4}{n-1} < s < n$. Connecting orbits are in blue and blowing up in red.

2.4 Case 4

Here we will consider the case $s > n \geq 2$. We will study solutions of the singular ODE

$$\frac{dy}{dx} = \frac{-(n-1)xy + \frac{s-n}{2}(1-y)^2}{x(1-x)}, \qquad 0 < x < 1. \tag{2.12}$$

At $x = 1$ the roots of the equation

$$-(n-1)y + \frac{s-n}{2}(1-y)^2 = 0, \qquad \Leftrightarrow \qquad y^2 - \frac{2(s-1)}{s-n}y + 1 = 0,$$

satisfy

$$0 < \tau_2 < 1 < \frac{s-1}{s-n} < \tau_1.$$

Here the important critical points of the ODE are

$$(0, 1), \qquad (1, \tau_1), \qquad (1, \tau_2)\ .$$

There are other critical points that lie on the curve

$$-(n-1)xy + \frac{n-s}{2}(1-y)^2 = 0, \qquad 0 < x < 1.$$

The region

$$-(n-1)xy + \frac{n-s}{2}(1-y)^2 < 0, \qquad 0 < x < 1,$$

is forward invariant, that is, if there is a pair (x_0, y_0) such that

$$-(n-1)x_0y_0 + \frac{n-s}{2}(1-y_0)^2 < 0, \qquad 0 < x_0 < 1,$$

then the solution $y(x)$ of (2.12) that passes through this point is defined for all $x \in (x_0, 1)$ and satisfies

$$-(n-1)xy(x) + \frac{n-s}{2}(1-y(x))^2 < 0, \qquad x_0 < x < 1.$$

We next have

Theorem 2.14. *Let $s > n \geq 2$. Then there is an analytic solution $y_a(x)$ of (2.12) near $(x, y) = (0, 1)$, that is defined for all $x \in (0, 1)$ and in addition* $\lim_{x\to 1^-} y_a(x) = \tau_2$.

Proof: The existence of an analytic solution near $(x, y) = (0, 1)$ follows in a similar manner as before. Near $x = 0$ it behaves as

$$y_a(x) = 1 - (n-1)x + O(x^2) \ .$$

As a consequence, for x near zero we have that

$$-(n-1)xy_a(x) + \frac{n-s}{2}(1-y_a(x))^2 < 0.$$

Using the forward invariant of the region we conclude that y_a is defined for all $0 < x < 1$ and that it decreases to τ_2.

$\square$

3 Proof of Theorems

In this section using the results of section 2 we will give the proofs of Theorems 1.1 and 1.2.

Proof of Theorem 1.1: If $\phi \in C^2(\bar{B}_1^c)$, $\phi > 0$ in $\bar{B}_1^c$ and $u \in C_c^\infty(\bar{B}_1^c)$ then by expanding

$$\int_{\bar{B}_1^c} \frac{1}{(|x|-1)^{s-2}} \left| \nabla u(x) - \frac{\nabla \phi(x)}{\phi(x)} u(x) \right|^2 dx \ ,$$

and integrating by parts, one concludes that

$$\int_{\bar{B}_1^c} \frac{|\nabla u(x)|^2}{(|x|-1)^{s-2}} \, dx \geq -\int_{\bar{B}_1^c} \frac{\nabla \cdot \left(\frac{\nabla \phi(x)}{(|x|-1)^{s-2}} \right)}{\phi} u^2 \, dx, \qquad \forall u \in C_c^\infty(\bar{B}_1^c). \tag{3.13}$$

In the sequel we will choose appropriate radial functions ϕ depending on s and n.

Case (a): Let either $n = 2, 3$ and $1 < s \leq \frac{n+1}{2}$ or else $n > 3$ and $1 < s < \frac{3n-5}{n-1}$. In this case we use the analytic function $y_a(x)$ given by Theorem (2.4) and set

$$\frac{2}{s-1}(r-1)\frac{\phi'(r)}{\phi(r)} = y_a(1/r), \qquad r > 1. \tag{3.14}$$

Solving this, we take

$$\phi(r) = (r-1)^{\frac{s-1}{2}} \ \exp\left[\frac{s-1}{2}\int_r^2 \frac{1-y_a(\frac{1}{\sigma})}{\sigma-1}d\sigma\right], \qquad r>1. \tag{3.15}$$

Due to the asymptotics of y_a near $x=1$ the limit

$$\lim_{r\to 1^+}\int_r^2 \frac{1-y_a(\frac{1}{\sigma})}{\sigma-1}d\sigma,$$

exists and is a finite number. As a consequence the following limit also exists

$$\lim_{r\to 1^+}\frac{\phi(r)}{(r-1)^{\frac{s-1}{2}}} = \exp\left[\frac{s-1}{2}\int_1^2 \frac{1-y_a(\frac{1}{\sigma})}{\sigma-1}d\sigma\right].$$

Due to the boundedness of $y_a(x)$ and (3.14) the following estimate is also true,

$$\left|\frac{\phi'(r)}{\phi(r)}\right| \le \frac{c}{r-1}, \qquad r>1; \tag{3.16}$$

we will use it later.

A straightforward calculation based on (3.15) and (2.1) shows that ϕ satisfies

$$\left(\frac{r^{n-1}\phi'(r)}{(r-1)^{s-2}}\right)' + \frac{\left(\frac{s-1}{2}\right)^2 r^{n-1}}{(r-1)^s}\,\phi(r) = 0, \qquad r>1. \tag{3.17}$$

Hence, using function ϕ in (3.13) we obtain that

$$c(n,s) = \left(\frac{s-1}{2}\right)^2,$$

in this case.

Case (b): Let either $n=2,3$ and $\frac{n+1}{2} < s < n$ or else $n>3$ and $\frac{n^2-3n+4}{n-1} \le s < n$. In this case we use the analytic function $y_a(x)$ given by Theorem (2.13) and set

$$\frac{2}{n-s}(r-1)\frac{\phi'(r)}{\phi(r)} = y_a(1/r), \qquad r>1. \tag{3.18}$$

Solving this, we take

$$\phi(r) = (r-1)^{-\frac{n-s}{2}} \ \exp\left[\frac{n-s}{2}\int_2^r \frac{1+y_a(\frac{1}{\sigma})}{\sigma-1}d\sigma\right], \qquad r>1. \tag{3.19}$$

Due to the asymptotics of y_a near $x=0$ the limit

$$\lim_{r\to\infty}\int_2^r \frac{1+y_a(\frac{1}{\sigma})}{\sigma-1}d\sigma,$$

exists and is a finite number. As a consequence the following limit also exists

$$\lim_{r\to+\infty}(r-1)^{\frac{n-s}{2}}\phi(r) = \exp\left[\frac{n-s}{2}\int_2^{+\infty} \frac{1+y_a(\frac{1}{\sigma})}{\sigma-1}d\sigma\right].$$

Due to the boundedness of $y_a(x)$ and (3.18) the following estimate is also true,

$$\left|\frac{\phi'(r)}{\phi(r)}\right| \leq \frac{c}{r-1}, \qquad r>1; \tag{3.20}$$

A straightforward calculation based on (3.19) and (2.10) shows that ϕ satisfies

$$\left(\frac{r^{n-1}\phi'(r)}{(r-1)^{s-2}}\right)' + \frac{\left(\frac{n-s}{2}\right)^2 r^{n-1}}{(r-1)^s}\phi(r) = 0, \qquad r>1. \tag{3.21}$$

Hence, using function ϕ in (3.13) we obtain that

$$c(n,s) = \left(\frac{n-s}{2}\right)^2.$$

Notice that in this case $\left(\frac{n-s}{2}\right)^2 < \left(\frac{s-1}{2}\right)^2$.

Case (c): Let $n>3$ and $\frac{3n-5}{n-1} < s < \frac{n^2-3n+4}{n-1}$.
We now take (cf Theorem 2.9)

$$\phi(r) = \frac{(r-1)^{\frac{(n-2)(s-2)}{n-3}}}{r^{n-2}}. \tag{3.22}$$

Again, it satisfies

$$\left|\frac{\phi'(r)}{\phi(r)}\right| \leq \frac{c}{r-1}, \qquad r>1; \tag{3.23}$$

and solves

$$\left(\frac{r^{n-1}\phi'(r)}{(r-1)^{s-2}}\right)' + \frac{\frac{(n-2)(n-s-1)(s-2)}{(n-3)^2}\, r^{n-1}}{(r-1)^s}\phi(r) = 0, \qquad r>1. \tag{3.24}$$

Using function ϕ in (3.13) we obtain

$$c(n,s) = \frac{(n-2)(n-s-1)(s-2)}{(n-3)^2}.$$

In this case $\frac{(n-2)(n-s-1)(s-2)}{(n-3)^2} < \min\{\left(\frac{s-1}{2}\right)^2, \left(\frac{n-s}{2}\right)^2\}$. We also note that ϕ is in the proper energy space and realizes the best constant, that is

$$\int_{\bar{B}_1^c} \frac{|\nabla\phi|^2}{(|x|-1)^{s-2}}\,dx = \frac{(n-2)(n-s-1)(s-2)}{(n-3)^2}\int_{\bar{B}_1^c}\frac{\phi^2}{(|x|-1)^s}\,dx.$$

Case (d): Let $s>n\geq 2$.
We similarly use the analytic function $y_a(x)$ of Theorem 2.14. We take

$$\phi(r) = (r-1)^{\frac{s-n}{2}}\ \exp\left[\frac{s-n}{2}\int_2^r \frac{y_a(\frac{1}{\sigma})-1}{\sigma-1}d\sigma\right], \qquad r>1. \tag{3.25}$$

From the asymptotics of $y_a(x)$ near $x=0$ we conclude that integral

$$\int_2^r \frac{y_a(\frac{1}{\sigma})-1}{\sigma-1}d\sigma,$$

exists and is a finite number. As a consequence the following limit also exists

$$\lim_{r\to+\infty}\frac{\phi(r)}{(r-1)^{\frac{s-n}{2}}}=\exp\left[\frac{n-s}{2}\int_2^{+\infty}\frac{y_a(\frac{1}{\sigma})-1}{\sigma-1}d\sigma\right].$$

Again, the following estimate is true,

$$\left|\frac{\phi'(r)}{\phi(r)}\right|\le\frac{c}{r-1},\qquad r>1; \tag{3.26}$$

Function ϕ solves

$$\left(\frac{r^{n-1}\phi'(r)}{(r-1)^{s-2}}\right)'+\frac{\left(\frac{n-s}{2}\right)^2r^{n-1}}{(r-1)^s}\phi(r)=0,\qquad r>1. \tag{3.27}$$

Hence, using function ϕ in (3.13) we obtain that

$$c(n,s)=\left(\frac{n-s}{2}\right)^2.$$

Conclusion of the proof. It remains to establish that $c(n,s)$ is not realized with the exception of case (c). Suppose on the contrary that there exists function ψ that satisfies

$$\int_{\bar{B}_1^c}\frac{|\nabla\psi|^2}{(|x|-1)^{s-2}}\,dx=c(n,s)\int_{\bar{B}_1^c}\frac{\psi^2}{(|x|-1)^s}\,dx.$$

Let $\phi(x)$ be the function defined by (3.15) in case (a), by (3.19) in case (b), and by (3.25) in case (d). In all cases we have $\left|\frac{\nabla\phi}{\phi}\right|\le\frac{c}{|x|-1}$ for all $|x|>1$ and the following calculations are justified

$$\begin{aligned}&\int_{\bar{B}_1^c}\frac{1}{(|x|-1)^{s-2}}\left|\nabla\psi-\frac{\nabla\phi}{\phi}\psi\right|^2\,dx\\ &=\int_{\bar{B}_1^c}\frac{|\nabla\psi|^2}{(|x|-1)^{s-2}}\,dx-\int_{\bar{B}_1^c}\frac{\nabla\phi\cdot\nabla\psi^2}{(|x|-1)^{s-2}\phi}\,dx+\int_{\bar{B}_1^c}\frac{|\nabla\phi|^2\psi^2}{(|x|-1)^{s-2}\phi^2}\,dx\\ &=\int_{\bar{B}_1^c}\frac{|\nabla\psi|^2}{(|x|-1)^{s-2}}\,dx+\int_{\bar{B}_1^c}\nabla\cdot\left(\frac{\nabla\phi}{(|x|-1)^{s-2}}\right)\frac{\psi^2}{\phi}\,dx\\ &=\int_{\bar{B}_1^c}\frac{|\nabla\psi|^2}{(|x|-1)^{s-2}}\,dx-c(n,s)\int_{\bar{B}_1^c}\frac{\psi^2}{(|x|-1)^s}\\ &=0.\end{aligned}$$

Hence, $\psi=k\phi$ which is a contradiction since ϕ is not in the energy space.

□

We also have

Proof of Theorem 1.2: Using the results of Theorem 1.1 and integrating in the y-variables we obtain the stated inequality. We next prove the optimality. From Theorem 1.1, given any $\varepsilon>0$ there exists $\phi\in C_c^\infty(\bar{B}_1^c)$ such that

$$c(n,s)\le\frac{\int_{\bar{B}_1^c}\frac{|\nabla\phi|^2}{(|x|-1)^{s-2}}dx}{\int_{\bar{B}_1^c}\frac{\phi^2}{(|x|-1)^s}dx}\le c(n,s)+\varepsilon.$$

We also consider a function $\psi \in C_c^\infty(B_R)$, $B_R \subset \mathbb{R}^m$. Then $\phi\psi \in C_c^\infty(B_1^c \times B_R)$ and

$$\begin{aligned}
\frac{\int_{\mathbb{R}^m}\int_{\bar{B}_1^c} \frac{|\nabla_{(x,y)}(\phi\psi)|^2}{(|x|-1)^{s-2}}\,dx\,dy}{\int_{\mathbb{R}^m}\int_{\bar{B}_1^c} \frac{(\phi\psi)^2}{(|x|-1)^{s}}\,dx\,dy} &= \frac{\int_{B_R}\int_{\bar{B}_1^c} \frac{|\nabla_x\phi(x)|^2\psi^2(y)+\phi^2(x)|\nabla_y\psi(y)|^2}{(|x|-1)^{s-2}}\,dx\,dy}{\int_{\bar{B}_1^c} \frac{\phi^2(x)}{(|x|-1)^{s}}\,dx \cdot \int_{B_R}\psi^2(y)\,dy} \\
&= \frac{\int_{\bar{B}_1^c} \frac{|\nabla\phi|^2}{(|x|-1)^{s-2}}dx}{\int_{\bar{B}_1^c} \frac{\phi^2}{(|x|-1)^{s}}dx} + \frac{\int_{\bar{B}_1^c} \frac{\phi^2}{(|x|-1)^{s-2}}dx}{\int_{\bar{B}_1^c} \frac{\phi^2}{(|x|-1)^{s}}dx} \cdot \frac{\int_{B_R}|\nabla_y\psi(y)|^2\,dy}{\int_{B_R}\psi^2(y)\,dy} \\
&< c(n,s) + 2\varepsilon,
\end{aligned}$$

by choosing R large enough and ψ close to the first Dirichlet eigenfunction in B_R. This establishes the optimality of $c(n,s)$.

It remains to show the non existence of minimizers. Suppose on the contrary that there exists $f(x,y)$ that satisfies

$$\int_{\mathbb{R}^m}\int_{\bar{B}_1^c} \frac{|\nabla_{(x,y)}f|^2}{(|x|-1)^{s-2}}\,dx\,dy = c(n,s)\int_{\mathbb{R}^m}\int_{\bar{B}_1^c} \frac{f^2}{(|x|-1)^{s}}\,dx\,dy.$$

In addition it solves the Euler–Lagrange equation, that is, for all $\psi \in C_c^\infty(B_1^c \times B_R)$, satisfies

$$\int_{\mathbb{R}^m}\int_{\bar{B}_1^c} \frac{\nabla_{(x,y)}f \cdot \nabla_{(x,y)}\psi}{(|x|-1)^{s-2}}\,dx\,dy = c(n,s)\int_{\mathbb{R}^m}\int_{\bar{B}_1^c} \frac{f\psi}{(|x|-1)^{s}}\,dx\,dy.$$

Let $\phi(x)$ be the function defined by (3.15) in case (a), by (3.19) in case (b), by (3.22) in case (c) and by (3.25) in case (d). In all cases we have $\left|\frac{\nabla\phi}{\phi}\right| \le \frac{c}{|x|-1}$ for all $|x|>1$ and the following calculations are justified

$$\begin{aligned}
&\int_{\mathbb{R}^m}\int_{\bar{B}_1^c} \frac{1}{(|x|-1)^{s-2}}\left|\nabla_{(x,y)}f - \frac{(\nabla_x\phi, 0)}{\phi}f\right|^2 \\
&= \int_{\mathbb{R}^m}\int_{\bar{B}_1^c} \frac{|\nabla_{(x,y)}f|^2}{(|x|-1)^{s-2}} - \int_{\mathbb{R}^m}\int_{\bar{B}_1^c} \frac{\nabla_x\phi\cdot\nabla_x f^2}{(|x|-1)^{s-2}\phi} + \int_{\mathbb{R}^m}\int_{\bar{B}_1^c} \frac{|\nabla_x\phi|^2 f^2}{(|x|-1)^{s-2}\phi^2} \\
&= \int_{\mathbb{R}^m}\int_{\bar{B}_1^c} \frac{|\nabla_{(x,y)}f|^2}{(|x|-1)^{s-2}} + \int_{\mathbb{R}^m}\int_{\bar{B}_1^c} \nabla_x\cdot\left(\frac{\nabla_x\phi}{(|x|-1)^{s-2}}\right)\frac{f^2}{\phi} \\
&= \int_{\mathbb{R}^m}\int_{\bar{B}_1^c} \frac{|\nabla_{(x,y)}f|^2}{(|x|-1)^{s-2}} - c(n,s)\int_{\mathbb{R}^m}\int_{\bar{B}_1^c} \frac{f^2}{(|x|-1)^{s}} \\
&= 0.
\end{aligned}$$

It follows that $f(x,y) = k\phi(x)$ for some constant k. However, since ϕ is independent of y it is not in the energy space in $\bar{B}_1^c \times \mathbb{R}^m$.

□

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