Improving L^2 estimates to Harnack inequalities

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Abstract

We consider operators of the form $\mathcal{L} = -L - V$, where L is an elliptic operator and V is a singular potential, defined on a smooth bounded domain $\Omega \subset I\!R^n$ with Dirichlet boundary conditions. We allow the boundary of Ω to be made of various pieces of different codimension. We assume that $\mathcal{L}$ has a generalized first eigenfunction of which we know two sided estimates. Under these assumptions we prove optimal Sobolev inequalities for the operator $\mathcal{L}$, we show that it generates an intrinsic ultracontractive semigroup and finally we derive a parabolic Harnack inequality up to the boundary as well as sharp heat kernel estimates.

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1 Introduction

Let $\Omega \subset I\!R^n$, be a bounded domain and suppose V is an $L^1_{loc}(\Omega)$ potential for which we know the following L^2 estimate

$$0 < \lambda_1 := \inf_{u \in C_0^\infty(\Omega)} \frac{\int_\Omega (|\nabla u|^2 - V(x)u^2)dx}{\int_\Omega u^2 dx} \ . \tag{1.1}$$

One of the motivations of the present work is whether one can improve the above estimate to a Sobolev type estimate, involving, if possible, the critical Sobolev exponent. It is clear that to improve the previous estimate one needs more information concerning the potential V, besides (1.1). One additional piece of information that we are going to use, is the existence of a generalized eigenfunction ϕ_1 of problem (1.1) as well as sharp two sided estimates of ϕ_1. Under this extra piece of information we are able to obtain sharp Sobolev type inequalities involving the critical Sobolev exponent.

The knowledge of the asymptotic behaviour of ϕ_1 usually comes as a consequence of the maximum principle and the local character of V. We present such an argument, following the ideas of Brezis, Marcus and Shafrir [5] for the critical potential $V(x) = \frac{1}{4}\frac{1}{\mathrm{dist}^2(x,\partial\Omega)}$ in the Appendix. We should mention that in this case the asymptotics of ϕ_1 have already been derived by Dávila and Dupaigne [12], [13]. The argument we present is much simpler and is based only on the maximum principle applied in the appropriate energy space. All the potentials we have in Section 4 as well as other potentials can be treated similarly.

Presupposing the existence and asymptotic behaviour of the generalized eigenfunction ϕ_1 seems to be a natural assumption. It is E. B Davies [8] who put forward the idea of connecting the asymptotics of ϕ_1 to the asymptotics of the Green function in the case of subcritical potentials V. In fact, he conjectured that knowing that the asymptotic behavior of ϕ_1 is like $d(x) := \mathrm{dist}(x, \partial\Omega)$ is actually equivalent to the exact two-sided Green function bounds. This conjecture was answered positively in [20] even for a larger class of potentials for which the generalized eigenfunction ϕ_1 behaves like $d^\alpha(x) := \mathrm{dist}^\alpha(x, \partial\Omega)$, for some $\alpha \geq 1/2$.

The idea of obtaining heat kernel estimates of second order elliptic operators with singular potentials in terms of the generalized ground state is not new and besides [23] and [24] has been successfully exploited in [26] and [27].

Another motivation is the study of the corresponding parabolic problem, especially when the potential V is singular (see e.g. [1], [6] and [31]). In connection with this, we mention the work of Cabré and Martel [6] where the condition $\lambda_1 > -\infty$ is shown to be necessary and sufficient for the existence of a global positive weak solution. They also show that these solutions grow at most exponentially in time for any nonnegative initial data $u_0 \in L^2(\Omega)$. In fact, as far as the parabolic problem is concerned, condition (1.1) is practically equivalent to $\lambda_1 > -\infty$ due to a shift in the time variable. Under the extra assumption that the asymptotics of the generalized eigenfunction are known, we show that the corresponding Schrödinger operator generates a semigroup of integral operators for which we obtain parabolic Harnack inequality up to the boundary and precise heat kernel estimates.

At this point we introduce some notations that we keep throughout the work. We assume that $\Omega \subset I\!\!R^n$, $n \geq 2$, is a bounded domain, with a boundary $\partial\Omega = \cup_{k=1}^{n}\Gamma_k$, where $\Gamma_k = \cup_{j=1}^{m_k}\Gamma_{k,j}$ is a finite union of m_k distinct smooth C^2 boundaryless hypersurfaces $\Gamma_{k,j}$ of codimension $\mathrm{codim}\Gamma_{k,j} = k$, where $j = 1, \ldots, m_k$, $k = 1, \ldots, n-1$; in addition $\Gamma_n = \{x_1, \ldots, x_{m_n}\}$, $\Gamma_{k,j} \cap \Gamma_{l,i} = \emptyset$ if $k \neq l$, or $i \neq j$ and $\Gamma_{k,j} \cap \Gamma_n = \emptyset$, for $k = 1, \ldots, n-1$ and $j = 1, \ldots, m_k$. We also set $d_k(x) = \mathrm{dist}(x, \Gamma_k)$. For $x \in \Omega$ we denote by $d(x)$ the distance to the boundary $\partial\Omega$. We clearly have that $d(x) = \min_{x\in\Omega}\{d_1(x), \ldots, d_n(x)\}$. Finally for the part of the boundary that is of codimension one we use the special notation $\partial_1\Omega = \Gamma_1$.

We are interested in the quadratic form

$$Q[u]=\int_\Omega\left(\sum_{i,j=1}^n a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}-Vu^2\right)dx, \qquad u\in C_0^\infty(\Omega), \tag{1.2}$$

where $V\in L^1_{loc}(\Omega)$ and $a_{ij}(x)$, $i,j=1,\dots,n$ is a measurable symmetric uniformly elliptic matrix, that is

$$C_0|\xi|^2\le\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j\le C_0^{-1}|\xi|^2, \qquad \xi\in I\!R^n \tag{1.3}$$

for some $C_0>0$. We also assume the L^2 estimate

$$0<\lambda_1:=\inf_{u\in C_0^\infty(\Omega)}\frac{Q[u]}{\int_\Omega u^2dx}, \tag{1.4}$$

and that to λ_1, there corresponds a generalized eigenfunction ϕ_1 of (1.4). More precisely, we assume that $\phi_1\in H^1_{loc}(\Omega)\cap L^\infty_{loc}(\Omega)$ and that

$$\int_\Omega\sum_{i,j=1}^n a_{ij}\frac{\partial\phi_1}{\partial x_i}\frac{\partial\psi}{\partial x_j}\,dx=\int_\Omega(V+\lambda_1)\phi_1\psi\,dx, \qquad \psi\in C_0^\infty(\Omega). \tag{1.5}$$

In addition we assume that we have two sided estimates on ϕ_1 of the form $\phi_1\sim d_1^{\alpha_1}\dots d_n^{\alpha_n}$, that is

$$c_1d_1^{\alpha_1}(x)\dots d_n^{\alpha_n}(x)\le\phi_1(x)\le c_2d_1^{\alpha_1}(x)\dots d_n^{\alpha_n}(x), \tag{1.6}$$

for any $x\in\Omega$, for two positive constants c_1,c_2 and for suitable exponents $\alpha_1,\dots,\alpha_n$. Appropriate conditions on the exponents α_i will be formulated below.

Our first result concerns the following improved Sobolev type inequality.

Theorem 1.1 (Optimal Sobolev type inequality) *For $V\in L^1_{loc}(\Omega)$ we assume that (1.4) holds and in addition there exists a ground state $\phi_1\in H^1_{loc}(\Omega)\cap L^\infty_{loc}(\Omega)$ satisfying (1.6) for*

$$\alpha_k>-\frac{k-2}{2}-(n-k)\frac{q-2}{2(q+2)}\ , \quad k=1,\dots,n\ , \tag{1.7}$$

where $2<q\le\frac{2n}{n-2}$ if $n\ge3$ and $q>2$ if $n=2$. We then have that

$$0<C(\Omega,\alpha_1,\dots,\alpha_n,q)=\inf_{u\in C_0^\infty(\Omega)}\frac{Q[u]}{\left(\int_\Omega d^{\frac{q(n-2)-2n}{2}}|u|^qdx\right)^{\frac{2}{q}}}\ . \tag{1.8}$$

In particular when $n\ge3$ and $q=\frac{2n}{n-2}$ we have that

$$0<C(\Omega,\alpha_1,\dots,\alpha_n)=\inf_{u\in C_0^\infty(\Omega)}\frac{Q[u]}{\left(\int_\Omega|u|^{\frac{2n}{n-2}}dx\right)^{\frac{n-2}{n}}}$$

if

$$\alpha_k>-\frac{k-2}{2}-\frac{(n-k)}{2(n-1)}, \qquad k=1,\dots,n\ .$$

We note that the condition (1.7) is optimal. In the extreme cases $q > 2$ and $\alpha_n = -\frac{n-2}{2}$ or $q = 2$ and $\alpha_k = -\frac{k-2}{2}$ we have different improved inequalities, see Theorems 2.4 and 2.5 respectively.

Using the above Sobolev type inequality we will now proceed to the study of the corresponding parabolic problem, that is

$$\begin{cases} \frac{\partial u}{\partial t} = -\mathcal{L}u := \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + V(x)u & \text{in} \quad (0,\infty)\times\Omega \ , \\ u(x,t) = 0 & \text{on} \quad (0,\infty)\times\partial_1\Omega, \\ u(x,0) = u_0(x) & \text{on} \quad \Omega \ . \end{cases} \tag{1.9}$$

Our first result is the following Harnack inequality.

Theorem 1.2 (Parabolic Harnack inequality up to the boundary) *For $V \in L^1_{loc}(\Omega)$ we assume (1.4) and that (1.6) holds for some $\alpha_k \geq -\frac{k-2}{2}$, for $k = 1, \cdots, n$. Then for $\mathcal{L}$ as in (1.9) there exist positive constants C_H and $R = R(\Omega)$ such that for $x \in \Omega$, $0 < r < R$ and for any positive solution $u(y,t)$ of*

$$\frac{\partial u}{\partial t} = -\mathcal{L}u \ \ in \ \ \{\mathcal{B}(x,r)\cap\Omega\}\times(0,r^2) \ ,$$

the following estimate holds true

$$\text{ess sup}_{(y,t)\in\left\{\mathcal{B}(x,\frac{r}{2})\cap\Omega\right\}\times(\frac{r^2}{4},\frac{r^2}{2})} \ u(y,t)\prod_{i=1}^{n} d_i^{-\alpha_i}(y) \leq$$

$$\leq C_H \ \text{ess inf}_{(y,t)\in\left\{\mathcal{B}(x,\frac{r}{2})\cap\Omega\right\}\times(\frac{3}{4}r^2,r^2)} \ u(y,t)\prod_{i=1}^{n} d_i^{-\alpha_i}(y).$$

Here $\mathcal{B}(x,r)$ denotes roughly speaking an n dimensional cube centered at x and having size r, see Definition 3.1 for details.

Theorem 1.2 states a parabolic Harnack inequality up to the boundary for the ratio of any positive local solution to the Cauchy-Dirichlet problem and the generalized eigenfunction ϕ_1. We note that $\alpha_1 \geq 1/2$, therefore ϕ_1 is zero on the boundary $\partial_1\Omega$. In particular it implies that any two nonnegative solutions vanishing on $\partial_1\Omega$ must vanish at the same rate. It is clear then that such a normalization is necessary. In fact the natural quantity is $v = u/\phi_1$ and it is for this function that we prove the Harnack inequality. For the definition of solution for the function v we refer to Definition 3.8, where however the appropriate weight is ϕ_1^2.

Alternatively, one could define local weak solutions of (1.9) using a suitable local energetic space obtained via the quadratic form (1.2). For an example of globally defined energetic solutions see [31].

Our result in the case $\alpha_1 = 1$, $\alpha_k = 0$ for $k = 2, \cdots, n$ is basically the local comparison principle of [14]. We also note that the restriction on α_k in Theorem 1.2 is sharp.

In what follows we denote by h the integral kernel of the L^2 semigroup associated to the elliptic operator $\mathcal{L}$ as defined in (1.9), that is

$$u(x,t) := \int_{\Omega} h(t,x,y)u_0(y)dy \ .$$

The existence of $h(t,x,y)$ is proved in Proposition 2.8 and it is a consequence of our Theorem 1.1.

As usual, from the parabolic Harnack inequality one can obtain sharp heat kernel estimates, as explained by Grigoryan and Saloff Coste, see [23], [24], [30]. In particular we have

Theorem 1.3 (Sharp heat kernel estimates) *For $V \in L^1_{loc}(\Omega)$ we assume (1.4) and that (1.6) holds for some $\alpha_k \geq -\frac{k-2}{2}$, for $k = 1, \cdots, n$. Then there exist positive constants C_1, C_2, with $C_1 \leq C_2$, and $T > 0$ depending on Ω such that*

$$C_1 \prod_{i=1}^{n} \left(1 + \frac{\sqrt{t}}{d_i(x)}\right)^{-\alpha_i} \left(1 + \frac{\sqrt{t}}{d_i(y)}\right)^{-\alpha_i} t^{-\frac{n}{2}} e^{-C_2 \frac{|x-y|^2}{t}} \leq h(t,x,y) \leq$$

$$\leq C_2 \prod_{i=1}^{n} \left(1 + \frac{\sqrt{t}}{d_i(x)}\right)^{-\alpha_i} \left(1 + \frac{\sqrt{t}}{d_i(y)}\right)^{-\alpha_i} t^{-\frac{n}{2}} e^{-C_1 \frac{|x-y|^2}{t}}$$

for all $x, y \in \Omega$ and $0 < t \leq T$, whereas

$$C_1 \prod_{i=1}^{n} d_i^{\alpha_i}(x)\ d_i^{\alpha_i}(y)\ e^{-\lambda_1 t} \leq h(t,x,y) \leq C_2 \prod_{i=1}^{n} d_i^{\alpha_i}(x)\ d_i^{\alpha_i}(y)\ e^{-\lambda_1 t}$$

for all $x, y \in \Omega$ and $t \geq T$.

Due to a shift in the time variable, Theorems 1.2 and 1.3 remain valid if we replace assumption (1.4) with the condition $\lambda_1 > -\infty$. For the corresponding statement of Theorem 1.1 under the condition $\lambda_1 > -\infty$ we refer to Theorem 2.1.

Although we present here only heat kernel estimates one can integrate in time to obtain the corresponding Green function estimates provided that $\lambda_1 > 0$.

It is clear that the asymptotics of ϕ_1 affect both the parabolic Harnack inequality and the heat kernel estimates, see Theorems 1.2 and 1.3. At the same time it seems that the Sobolev inequality is independent of the α_k's, in the sense that the exponents α_k do not appear in the ratio (1.8). We note however that there are critical cases where relation (1.8) fails and different Sobolev inequalities hold true. For instance if $\alpha_n = -\frac{n-2}{n}$ estimate (1.8) is no longer true; instead, the optimal Sobolev inequality involves a logarithmic correction see inequality (2.23) in Theorem 2.4. For other examples see Theorem A' in [22].

We note that instead of the uniform ellipticity condition (1.3) which we assume throughout this work, our method can also treat degenerate operators for which the following condition holds

$$C_0 w(x)|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq C_0^{-1} w(x)|\xi|^2, \qquad \xi \in I\!R^n\ , \tag{1.10}$$

where $w(x)$ is like a power of the distance function.

Finally we should mention that this work complements and extends our previous work [20]. There we studied the cases where the potential $V(x)$ is either $\frac{(n-2)^2}{4|x|^2}$ for a general bounded domain Ω or else $\frac{1}{4\mathrm{dist}^2(x,\partial\Omega)}$ for a convex bounded domain Ω. In the second case the convexity was used in an essential way. Here, even in these two cases, we improve our results in the first case by allowing potentials involving distances to more than one point and in the second case by removing the convexity assumption (see Section 4).

The article is organized as follows. In Section 2 we establish a Sobolev type inequality, thus giving the proof of Theorem 1.1, starting from the L^2 estimate and using the behavior of the generalized eigenfunction ϕ_1. In Section 3 we study the associated Cauchy-Dirichlet problem and prove a parabolic Harnack inequality up to the boundary as well as sharp two sided estimates on the corresponding heat kernel. In particular we provide the proof of Theorems 1.2 and 1.3. Finally, in Section 4 we give some examples of concrete Schrödinger operators with singular potentials for which an L^2 inequality holds true and the behavior of ϕ_1 is known. In all these examples the results of the present work apply.

2 From the L^2 estimate to Sobolev type inequalities

In this section starting from the L^2 estimate (1.4) we will prove various Sobolev inequalities involving optimal exponents. In particular we will prove a weighted logarithmic Sobolev inequality that will be crucial in establishing the intrinsic ultracontractivity of the semigroup associated with the operator $\mathcal{L}$ defined in (1.9).

For δ small enough we set

$$\Gamma_k^\delta = \{x \in \Omega, \quad \text{s.t. } \operatorname{dist}(x, \Gamma_k) < \delta\} \quad \text{and} \quad (\Gamma_k^\delta)^c = \Omega \setminus \Gamma_k^\delta.$$

As a consequence of the assumptions on the domain Ω we made in the introduction, we have that for δ small enough $\Gamma_{k,j}^\delta \cap \Gamma_{l,i}^\delta = \emptyset$ if $k \neq l$, or $i \neq j$ and $\Gamma_{k,j}^\delta \cap \Gamma_n^\delta = \emptyset$, for $k = 1, \ldots, n-1$ and $j = 1, \ldots, m_k$. We note that $\Omega = \left(\cup_{k=1}^n \Gamma_k^\delta\right) \cup \left(\cup_{k=1}^n \Gamma_k^\delta\right)^c = \left(\cup_{k=1}^n \Gamma_k^\delta\right) \cup \left(\cap_{k=1}^n (\Gamma_k^\delta)^c\right)$.

We are now ready to give the proof of Theorem 1.1. Indeed we will more generally assume instead of (1.4) the following L^2 estimate

$$-\infty < \lambda_1 := \inf_{u \in C_0^\infty(\Omega)} \frac{Q[u]}{\int_\Omega u^2 dx}, \tag{2.1}$$

Then we prove:

Theorem 2.1 (Optimal Sobolev type inequality) *For $V \in L^1_{loc}(\Omega)$ we assume that (2.1) holds and in addition there exists a ground state $\phi_1 \in H^1_{loc}(\Omega) \cap L^\infty_{loc}(\Omega)$ satisfying (1.6) for*

$$\alpha_k > -\frac{k-2}{2} - (n-k)\frac{q-2}{2(q+2)} \ , \quad k = 1, \ldots, n \tag{2.2}$$

where $2 < q \leq \frac{2n}{n-2}$ if $n \geq 3$ and $q > 2$ if $n = 2$. We then have that for every $\lambda > 0$

$$0 < C(\Omega, \alpha_1, \ldots, \alpha_n, q, \lambda) = \inf_{u \in C_0^\infty(\Omega)} \frac{Q[u] + (\lambda - \lambda_1) \int_\Omega u^2 dx}{\left(\int_\Omega d^{\frac{q(n-2)-2n}{2}} |u|^q dx\right)^{\frac{2}{q}}} \ . \tag{2.3}$$

In particular when $n \geq 3$ and $q = \frac{2n}{n-2}$ we have that for every $\lambda > 0$,

$$0 < C(\Omega, \alpha_1, \ldots, \alpha_n, \lambda) = \inf_{u \in C_0^\infty(\Omega)} \frac{Q[u] + (\lambda - \lambda_1) \int_\Omega u^2 dx}{\left(\int_\Omega |u|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}}},$$

if

$$\alpha_k > -\frac{k-2}{2} - \frac{(n-k)}{2(n-1)}, \qquad k = 1, \ldots, n \; .$$

In the case $\lambda_1 > 0$ one can take $\lambda = \lambda_1$, thus proving Theorem 1.1.

Proof of Theorem 2.1: It is a consequence of the following estimate for any $v \in C_0^\infty(\Omega)$

$$\int_\Omega \phi_1^2 \left(\sum_{i,j=1}^n a_{ij} v_{x_i} v_{x_j} + \lambda v^2 \right) dx \geq C \left(\int_\Omega \phi_1^q d^{\frac{q(n-2)-2n}{2}} |v|^q dx \right)^{\frac{2}{q}}, \tag{2.4}$$

with $C = C(\alpha_1, \ldots, \alpha_n, \Omega, \lambda) > 0$. For convenience we write v_{x_i} instead of $\frac{\partial v}{\partial x_i}$. Let us accept (2.4) and give the proof of the Theorem. Clearly (2.4) is valid not only for smooth functions but also for functions in the completion of $C_0^\infty(\Omega)$ under the norm defined by

$$||v||_{H^1_{\phi_1}} := \left(\int_\Omega \phi_1^2 (v^2 + |\nabla v|^2) dx \right)^{1/2} \; . \tag{2.5}$$

In particular we can take $v = \frac{u}{\phi_1}$, with $u \in C_0^\infty(\Omega)$, in which case we get

$$\sum_{i,j}^n \int_\Omega a_{ij} u_{x_i} u_{x_j} dx - 2 \sum_{i,j}^n \int_\Omega a_{ij} u u_{x_i} \frac{(\phi_1)_{x_j}}{\phi_1} dx + \sum_{i,j}^n \int_\Omega a_{ij} \frac{(\phi_1)_{x_i} (\phi_1)_{x_j}}{\phi_1^2} u^2 dx$$
$$+ \lambda \int_\Omega u^2 dx \geq C \left(\int_\Omega d^{\frac{q(n-2)-2n}{2}} |u|^q dx \right)^{\frac{2}{q}}. \tag{2.6}$$

On the other hand, by standard approximation arguments, (1.5) is valid also for $\psi = \frac{u^2}{\phi_1}$, with $u \in C_0^\infty(\Omega)$. For this choice of the test function, we get that

$$2 \sum_{i,j}^n \int_\Omega a_{ij} u u_{x_i} \frac{(\phi_1)_{x_j}}{\phi_1} dx - \sum_{i,j}^n \int_\Omega a_{ij} \frac{(\phi_1)_{x_i} (\phi_1)_{x_j}}{\phi_1^2} u^2 dx = \int_\Omega (V + \lambda_1) u^2 dx. \tag{2.7}$$

Combining (2.6) and (2.7) we conclude that for any $u \in C_0^\infty(\Omega)$,

$$Q[u] + (\lambda - \lambda_1) \int_\Omega u^2 dx \geq C \left(\int_\Omega d^{\frac{q(n-2)-2n}{2}} |u|^q dx \right)^{\frac{2}{q}},$$

which is the same as (2.3).

It remains to prove (2.4). Because of the ellipticity condition (1.3), estimate (2.4) follows from

$$\int_\Omega \phi_1^2 \left(|\nabla v|^2 + v^2 \right) dx \geq C \left(\int_\Omega \phi_1^q d^{\frac{q(n-2)-2n}{2}} |v|^q dx \right)^{\frac{2}{q}} \; , \; v \in C_0^\infty(\Omega) \; . \tag{2.8}$$

In view of (1.6) we may replace ϕ_1 in (2.8) by $d_1^{\alpha_1} \ldots d_n^{\alpha_n}$. Estimate (2.8) then is true in $\cap_{k=1}^n (\Gamma_k^\delta)^c$. This is a consequence of the standard Sobolev imbedding of functions in

$H^1(\cap_{k=1}^n(\Gamma_k^\delta)^c)$ in $L^q(\cap_{k=1}^n(\Gamma_k^\delta)^c)$ and the fact that $\delta \le d_k(x) \le D_k := \sup_{x\in\Omega} d_k(x) < \infty$, $k = 1, \dots, n$.

We therefore need to prove that estimate (2.8) is true when we replace Ω by $\cup_{k=1}^n \Gamma_k^\delta$. As a matter of fact it is enough to prove that for any $k = 1, \dots, n$,

$$\int_{\Gamma_k^\delta} d_k^{2\alpha_k}(|\nabla v|^2 + v^2)dx \ge C\left(\int_{\Gamma_k^\delta} d_k^{\beta_k q}|v|^q dx\right)^{\frac{2}{q}}, \qquad v \in C_0^\infty(\Omega),$$

where $\beta_k = \alpha_k - 1 + \frac{q-2}{2q}n$. The validity of this estimate is given in the next main Lemma.

Lemma 2.2 *Let $\Omega \subset I\!R^n$, $n \ge 2$, be a bounded domain. Suppose that $\Gamma_k \subseteq \partial\Omega$ is a smooth boundaryless hypersurface of codimension k, $k = 1, \dots, n-1$. When $k = n$ we take Γ_n to be a point. We also assume that*

$$\beta_k = \alpha_k - 1 + \frac{q-2}{2q}n \tag{2.9}$$

where $2 < q \le \frac{2n}{n-2}$ if $n \ge 3$, $q > 2$ if $n = 2$. Then, for any $k = 1, \dots, n$, there exists a $C = C(\alpha_k, \Omega, \delta, k) > 0$ such that for all $\delta > 0$ and all $v \in C_0^\infty(\Omega)$ there holds

$$\int_{\Gamma_k^\delta} d_k^{2\alpha_k}(|\nabla v|^2 + v^2)dx \ge C\|d_k^{\beta_k} v\|^2_{L^q(\Gamma_k^\delta)}, \tag{2.10}$$

provided that

$$\alpha_k \ne -\frac{k-2}{2} - (n-k)\frac{q-2}{2(q+2)}. \tag{2.11}$$

Proof: Let us fix a $k = 1, \dots, n$. We will initially establish the result for δ small. For simplicity we write d instead of d_k. From Lemma 4.2 [19] (see also [18]), we have that if

$$1 < Q \le \frac{n}{n-1}, \quad b = a - 1 + \frac{Q-1}{Q}n, \quad \text{and} \quad a \ne 1 - k, \tag{2.12}$$

then, for δ small there exists a $C > 0$ such that there holds

$$C\|d^b w\|_{L^Q(\Gamma_k^\delta)} \le \int_{\Gamma_k^\delta} d^a|\nabla w|dx + \int_{\partial\Gamma_k^\delta} d^a|w|dS_x, \qquad w \in C_0^\infty(\Omega \setminus \Gamma_k). \tag{2.13}$$

We apply (2.13) to the function $w = |v|^s$, $s = \frac{2+q}{2}$. Also, for α_k, β_k, and q as in (2.9), we set

$$Q = qs^{-1}, \quad b = \beta_k s, \quad a = b + 1 - \frac{Q-1}{Q}n = \frac{\beta_k q}{2} + \alpha_k.$$

It is easy to check that a, b, Q thus defined satisfy (2.12). As far as the condition $a \ne 1 - k$ is concerned, when written in terms of α_k, q, k and n, it is equivalent to

$$\alpha_k \ne -\frac{k-2}{2} - (n-k)\frac{q-2}{2(q+2)},$$

which is precisely (2.11). From (2.13) we have

$$C\|d^{\beta_k} v\|^{1+\frac{q}{2}}_{L^q(\Gamma_k^\delta)} = C\|d^b|v|^s\|_{L^Q(\Gamma_k^\delta)} \le s\int_{\Gamma_k^\delta} d^a|v|^{s-1}|\nabla v|dx + \int_{\partial\Gamma_k^\delta} d^a|v|^s dS_x, \tag{2.14}$$

for some positive constant C. Using Holder's inequality in the gradient term of the right hand side we get

$$\begin{aligned}
\int_{\Gamma_k^\delta} d^a|v|^{s-1}|\nabla v|dx &= \int_{\Gamma_k^\delta} d^{\alpha_k}|\nabla v| \quad d^{\frac{\beta_k q}{2}}|v|^{\frac{q}{2}}dx \\
&\leq \|d^{\alpha_k}|\nabla v|\|_{L^2(\Gamma_k^\delta)} \quad \|d^{\beta_k}v\|_{L^q(\Gamma_k^\delta)}^{\frac{q}{2}} \\
&\leq c_\varepsilon\|d^{\alpha_k}|\nabla v|\|_{L^2(\Gamma_k^\delta)}^{1+\frac{q}{2}} + \varepsilon\|d^{\beta_k}v\|_{L^q(\Gamma_k^\delta)}^{1+\frac{q}{2}}. \qquad (2.15)
\end{aligned}$$

Hence, from (2.14) and (2.15) we arrive at

$$(C-\varepsilon s)\|d^{\beta_k}v\|_{L^q(\Gamma_k^\delta)}^{1+\frac{q}{2}} \leq sc_\varepsilon\|d^{\alpha_k}|\nabla v|\|_{L^2(\Gamma_k^\delta)}^{1+\frac{q}{2}} + \int_{\partial\Gamma_k^\delta} d^a|v|^s dS_x. \qquad (2.16)$$

To continue we will estimate the trace term in (2.16). Using Holder's inequality we have that

$$\begin{aligned}
\int_{\partial\Gamma_k^\delta} d^a|v|^s dS_x &= \delta^a\int_{\partial\Gamma_k^\delta}|v|^{\frac{2+q}{2}}dx \leq c\,\delta^a\left(\int_{\partial\Gamma_k^\delta}|v|^{\frac{2(n-1)}{n-2}}dx\right)^{\frac{(n-2)(2+q)}{4(n-1)}} \\
&= c\,\delta^{\frac{(\beta_k-\alpha_k)q}{2}}\|d^{\alpha_k}v\|_{L^{\frac{2(n-1)}{n-2}}(\partial\Gamma_k^\delta)}^{1+\frac{q}{2}}
\end{aligned} \qquad (2.17)$$

By the trace imbedding [3], we have that for $u \in H^1(\Gamma_k^\delta)$

$$\|u\|_{L^{\frac{2(n-1)}{n-2}}(\partial\Gamma_k^\delta)}^2 \leq C(n,k)\|\nabla u\|_{L^2(\Gamma_k^\delta)}^2 + M\|u\|_{L^2(\Gamma_k^\delta)}^2, \qquad (2.18)$$

where $M = M(n,\Gamma_k^\delta)$. Applying this to $u = d^{\alpha_k}v$ we get

$$\|d^{\alpha_k}v\|_{L^{\frac{2(n-1)}{n-2}}(\partial\Gamma_k^\delta)}^2 \leq C_2\int_{\Gamma_k^\delta} d^{2\alpha_k}v^2dx + C_2\int_{\Gamma_k^\delta} d^{2\alpha_k}|\nabla v|^2dx, \qquad (2.19)$$

with $C_2 = C_2(\alpha_k,\delta,k,n)$.

Indeed after some elementary calculations from (2.18) we get for any $\theta > 1$

$$\begin{aligned}
\|d^{\alpha_k+\theta}v\|_{L^{\frac{2(n-1)}{n-2}}(\partial\Gamma_k^\delta)}^2 &\leq 2C(n,k)(\alpha_k+\theta)^2\int_{\Gamma_k^\delta} d^{2(\alpha_k+\theta)-2}v^2dx \\
&\quad +2C(n,k)\int_{\Gamma_k^\delta} d^{2(\alpha_k+\theta)}|\nabla v|^2dx + M\int_{\Gamma_k^\delta} d^{2(\alpha_k+\theta)}v^2dx \\
&\leq (2C(n,k)(\alpha_k+\theta)^2\delta^{2\theta-2} + M\delta^{2\theta})\int_{\Gamma_k^\delta} d^{2\alpha_k}v^2dx \\
&\quad +2C(n,k)\delta^{2\theta}\int_{\Gamma_k^\delta} d^{2\alpha_k}|\nabla v|^2dx.
\end{aligned}$$

Whence,

$$\|d^{\alpha_k}v\|_{L^{\frac{2(n-1)}{n-2}}(\partial\Gamma_k^\delta)}^2 \leq (2C(n,k)(\alpha_k+\theta)^2\delta^{-2}+M)\int_{\Gamma_k^\delta} d^{2\alpha_k}v^2dx + 2C(n,k)\int_{\Gamma_k^\delta} d^{2\alpha_k}|\nabla v|^2dx, \qquad (2.20)$$

and therefore (2.19). We note that (2.19) is valid even for nonpositive values of α_k.

Combining (2.16), (2.17), (2.19) and then raising to the power $\frac{4}{2+q}$ we easily conclude (2.10) for δ small.

The general case follows by noticing that outside Γ_k^δ for small δ the corresponding estimate comes from the standard Sobolev embeddings and the fact that $\delta \le d_k(x) \le D_k := \sup_{x\in\Omega} d_k(x) < \infty$, $k = 1, \ldots, n$.

This completes the proof of the Lemma as well as of Theorem 2.1.

$\square$

We note that when $k = n$ condition (2.11) reads $\alpha_n \neq -\frac{n-2}{2}$. It turns out that the analogue of the estimate (2.10) in case $k = n$ and $\alpha_n = -\frac{n-2}{2}$, involves logarithmic corrections. More precisely we have:

Lemma 2.3 *Let $\Omega \subset I\!R^n$, $n \geq 3$, be a bounded domain and $\Gamma_n = \{x_0\} \subset \partial\Omega$ be a point, such that for some δ_0 small $\Gamma_n^{\delta_0} = B_{\delta_0}(x_0) \setminus \{x_0\} \subset \Omega$. We also assume that*

$$\alpha_n = -\frac{n-2}{2}, \qquad 2 < q \le \frac{2n}{n-2}, \qquad \beta_n = -\frac{n-2}{2} - 1 + \frac{q-2}{2q}n.$$

Then, there exists a $C = C(\Omega, \delta, n) > 0$ such that for all $\delta > 0$ and all $v \in C_0^\infty(\Omega)$ there holds

$$\int_{\Gamma_n^\delta} d_n^{2-n}(|\nabla v|^2 + v^2)dx \geq C\|d^{\beta_n} X^{\frac{1}{2}+\frac{1}{q}} v\|^2_{L^q(\Gamma_n^\delta)}, \tag{2.21}$$

where $X = X(\frac{d_n(x)}{D_n})$, with $X(t) = (1-\ln t)^{-1}$, $0 < t \le 1$, and $D_n := \sup_{x\in\Omega} d_n(x) < \infty$.

Proof: As in the previous Lemma it is enough to give the proof for δ small. We may assume that $x_0 = 0$, hence, $d_n(x) = |x|$. Also, for simplicity we suppose that $\delta = 1$. Then we recall the following result for any $w \in C_0^\infty(B_2)$ the following estimate holds

$$\int_{B_2} |x|^{2-n}|\nabla w|^2 dx \geq c\left(\int_{B_2} |x|^{\beta_n q} X^{1+\frac{q}{2}}|w|^q dx\right)^{\frac{2}{q}} = c\|d^{\beta_n} X^{\frac{1}{2}+\frac{1}{q}} w\|^2_{L^q(B_2)}. \tag{2.22}$$

This is Lemma 3.2 [22] in the case $q = \frac{2n}{n-2}$ and Proposition 6.2 [2] in the case where $2 < q \le \frac{2n}{n-2}$. Given a function $v \in C_0^\infty(\Omega)$, we extent it from B_1 to the function $\tilde{v}$ supported in B_2 so that

$$\|\tilde{v}\|_{B_2} \le C\|v\|_{B_1},$$

holds for a positive constant depending only on n, where we denote by $\|v\|^2_{B_1} := \int_{B_1} |x|^{2-n}(|\nabla v|^2 + |v|^2)dx$ (note that away from the origin v is an H^1 function). We next apply (2.22) to $\tilde{v}$ and the result follows easily. We note that one cannot take a smaller exponent of the logarithmic term X.

$\square$

As a consequence of the above Lemma we have:

Theorem 2.4 *For $V \in L^1_{loc}(\Omega)$ we assume that (2.1) holds and in addition there exists a ground state $\phi_1 \in H^1_{loc}(\Omega) \cap L^\infty_{loc}(\Omega)$ satisfying (1.6) for $k = 1, \ldots, n-1$; $n \geq 3$*

$$\alpha_k > -\frac{k-2}{2} - (n-k)\frac{q-2}{2(q+2)}, \qquad \alpha_n = -\frac{n-2}{2}, \qquad 2 < q \le \frac{2n}{n-2}.$$

We then have that for every $\lambda > 0$

$$0 < C(\Omega, \alpha_1, \ldots, \alpha_n, q, \lambda) = \inf_{u \in C_0^\infty(\Omega)} \frac{Q[u] + (\lambda - \lambda_1) \int_\Omega u^2 dx}{\left(\int_\Omega d^{\frac{q(n-2)-2n}{2}} X^{\frac{q}{2}+1} |u|^q dx \right)^{\frac{2}{q}}}, \tag{2.23}$$

where $X = X(\frac{d_n(x)}{D_n})$, with $X(t) = (1 - \ln t)^{-1}$, $0 < t \leq 1$ and $D_n := \sup_{x \in \Omega} d_n(x) < \infty$. In particular by choosing $q = \frac{2n}{n-2}$ we have that for every $\lambda > 0$

$$0 < C(\Omega, \alpha_1, \ldots, \alpha_n, \lambda) = \inf_{u \in C_0^\infty(\Omega)} \frac{Q[u] + (\lambda - \lambda_1) \int_\Omega u^2 dx}{\left(\int_\Omega X^{\frac{2(n-1)}{n-2}} |u|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}}}.$$

Proof: The proof is quite similar to the proof of Theorem 2.1, where in the place of Lemma 2.2 one uses Lemma 2.3 for $k = n$. We omit further details.

□

Concerning the limit case $q = 2$ and $\alpha_k = -\frac{k-2}{2}$ we have the following

Theorem 2.5 *For $V \in L^1_{loc}(\Omega)$ we assume that (2.1) holds and in addition there exists a ground state $\phi_1 \in H^1_{loc}(\Omega) \cap L^\infty_{loc}(\Omega)$ satisfying (1.6) for $\alpha_k \geq -\frac{k-2}{2}$, $k = 1, \ldots, n$; $n \geq 2$. We then have that for every $\lambda > 0$*

$$0 < C(\Omega, \alpha_1, \ldots, \alpha_n, \lambda) = \inf_{u \in C_0^\infty(\Omega)} \frac{Q[u] + (\lambda - \lambda_1) \int_\Omega u^2 dx}{\int_\Omega X^2 \frac{u^2}{d^2} dx},$$

where $X = X(\frac{d(x)}{D})$, with $X(t) = (1 - \ln t)^{-1}$, $0 < t \leq 1$, and $D := \sup_{x \in \Omega} d(x) < \infty$.

Proof: The proof is quite similar to the proof of Theorem 2.1, where in the place of Lemma 2.2 one uses Lemma 2.6 below. We omit further details.

□

Lemma 2.6 *Let $\Omega \subset I\!R^n$, $n \geq 2$, be a bounded domain. Suppose that $\Gamma_k \subseteq \partial\Omega$ is a smooth boundaryless hypersurface of codimension k, $k = 1, \ldots, n-1$. When $k = n$ we take Γ_n to be a point. Then, for any $k = 1, \ldots, n$, there exists a $C > 0$ such that for all $\delta > 0$ and all $v \in C_0^\infty(\Omega)$ there holds*

$$\int_{\Gamma_k^\delta} d_k^{-(k-2)} (|\nabla v|^2 + v^2) dx \geq C \int_{\Gamma_k^\delta} X^2 d_k^{-k} v^2 dx,$$

where $X = X(\frac{d_k(x)}{D_k})$, with $X(t) = (1 - \ln t)^{-1}$, $0 < t \leq 1$, and $D_k := \sup_{x \in \Omega} d_k(x) < \infty$.

Proof: This is proved using similar ideas as in Lemma 2.3. We omit further details.

□

As a consequence of Theorems 2.1 and 2.4 we obtain

Theorem 2.7 (Weighted log Sobolev) *For $V \in L^1_{loc}(\Omega)$ we assume that (2.1) holds and in addition there exists a ground state $\phi_1 \in H^1_{loc}(\Omega) \cap L^\infty_{loc}(\Omega)$ satisfying (1.6) for $n \geq 2$ with*

$$\alpha_k > -\frac{k-2}{2} - \frac{n-k}{2(n-1)}, \qquad k = 1, \ldots, n-1, \qquad \alpha_n \geq -\frac{n-2}{2}. \tag{2.24}$$

Let

$$A := max\{\alpha_1, \alpha_2, \ldots, \alpha_n, 0\}.$$

Then, there exists a positive constant K *such that for any* ε *positive and for any* $u \in C_0^\infty(\Omega)$ *there holds*

$$\int_\Omega u^2 \ln\left(\frac{|u|}{\|u\|_2 d_1^{\alpha_1} \ldots d_n^{\alpha_n}}\right) dx \leq \varepsilon Q[u] + \left(K - \frac{n+2A}{4}\ln\varepsilon\right)\|u\|_2^2; \tag{2.25}$$

here $\|u\|_2^2 = \int_\Omega |u|^2 dx$.

Proof: At first we will show that in each Γ_k^δ, the following estimate holds:

$$\int_{\Gamma_k^\delta} \phi_1^2 v^2 \ln\left(\frac{|v|}{\|v\|_{\Gamma_k}}\right) dx \leq \varepsilon \int_{\Gamma_k^\delta} \phi_1^2(|\nabla v|^2 + v^2)dx + \left(K - \frac{n+2\alpha_k^+}{4}\ln\varepsilon\right)\|v\|_{\Gamma_k}^2, \tag{2.26}$$

where $\alpha_k^+ := \max\{\alpha_k, 0\}$ and $\|v\|_{\Gamma_k}^2 := \int_{\Gamma_k^\delta} \phi_1^2 v^2 dx$. To this end, let us assume first that w is normalized so that for $d\mu = \phi_1^2 w^2 dx$ one has $\int_{\Gamma_k^\delta} d\mu = \|w\|_{\Gamma_k}^2 = 1$. Then, for $q > 2$, using Jensen's inequality, we have

$$\int_{\Gamma_k^\delta} \phi_1^2 w^2 \ln|w|\, dx = \frac{1}{q-2}\int_{\Gamma_k^\delta} \ln|w|^{q-2} d\mu \leq \frac{q}{2(q-2)}\ln\left(\int_{\Gamma_k^\delta} \phi_1^2 |w|^q dx\right)^{\frac{2}{q}}. \tag{2.27}$$

To continue, we will use the estimate

$$\left(\int_{\Gamma_k^\delta} \phi_1^2 |w|^q dx\right)^{\frac{2}{q}} \leq C\int_{\Gamma_k^\delta} \phi_1^2(|\nabla w|^2 + w^2)dx. \tag{2.28}$$

In case $k = 1, \ldots, n-1$ or $k = n$ and $\alpha_n > -\frac{n-2}{2}$, (2.28) is a direct consequence of (2.10), provided that $d_k^{2\alpha_k} \leq c d_k^{q\beta_k}$, that is, $2\alpha_k \geq q\beta_k$. In view of the definition of β_k (see (2.9)), the requirement $2\alpha_k \geq q\beta_k$ is equivalent to

$$q(n-2+2\alpha_k) \leq 2(n+2\alpha_k).$$

We note that when $n \geq 3$ if $\alpha_k \leq 0$, then we can choose $q = \frac{2n}{n-2}$, whereas if $\alpha_k > 0$, then the maximum q one can choose is $q = \frac{2(n+2\alpha_k)}{(n-2+2\alpha_k)} < \frac{2n}{n-2}$. The same choice of q is feasible when $n = 2$ and $\alpha_1, \alpha_2 > 0$. Hence, in any case one takes

$$q = \frac{2(n+2\alpha_k^+)}{(n-2+2\alpha_k^+)}, \tag{2.29}$$

On the other hand, in case $k = n$ and $\alpha_n = -\frac{n-2}{2}$, estimate (2.28) is a direct consequence of (2.21) with $q = \frac{2n}{n-2}$ if $n \geq 3$; indeed, in this case one has $q\beta_n = -n$ and clearly $d_n^{2-n} \leq c d_n^{-n} X^{\frac{q}{2}+1}$. In particular, in all cases the choice of q is given by (2.29).

From (2.27) and (2.28) we get that

$$\int_{\Gamma_k^\delta} \phi_1^2 w^2 \ln|w|\, dx \leq \frac{q}{2(q-2)}\ln\left(C\int_{\Gamma_k^\delta} \phi_1^2(|\nabla w|^2 + w^2)dx\right).$$

Using the fact that $\ln\theta \le \varepsilon\theta - \ln\varepsilon$ for all θ, ε positive we get that there exists a $K > 0$ such that for any $\varepsilon > 0$

$$\int_{\Gamma_k^\delta} \phi_1^2 w^2 \ln|w|\, dx \le \varepsilon \int_{\Gamma_k^\delta} \phi_1^2(|\nabla w|^2 + w^2)dx + K - \frac{q}{2(q-2)}\ln\varepsilon. \tag{2.30}$$

Because of (2.29) we have that

$$\frac{q}{2(q-2)} = \frac{n+2\alpha_k^+}{4}.$$

On the other hand given any $v \in C_0^\infty(\Omega)$ we apply (2.30) to $w = \frac{v}{\|v\|_{\Gamma_k}}$ to conclude (2.26).

We next consider a $w \in C_0^\infty(\Omega)$ normalized by $\|w\|_\Omega^2 := \int_\Omega \phi_1^2 w^2 dx = 1$. Applying (2.26) to this w we get

$$\begin{aligned}\int_{\Gamma_k^\delta} \phi_1^2 w^2 \ln|w| dx \quad &- \quad \int_{\Gamma_k^\delta} \phi_1^2 w^2 \ln(\|w\|_{\Gamma_k})dx \le \varepsilon \int_{\Gamma_k^\delta} \phi_1^2(|\nabla w|^2 + w^2)dx \\ &+ \quad \left(K - \frac{n+2\alpha_k^+}{4}\ln\varepsilon\right)\|w\|_{\Gamma_k}^2.\end{aligned}$$

Since $\|w\|_{\Gamma_k} \le 1$ and therefore $\ln(\|w\|_{\Gamma_k}) \le 0$, we have in particular that

$$\int_{\Gamma_k^\delta} \phi_1^2 w^2 \ln|w| dx \le \varepsilon \int_{\Gamma_k^\delta} \phi_1^2(|\nabla w|^2 + w^2)dx + \left(K - \frac{n+2\alpha_k^+}{4}\ln\varepsilon\right)\|w\|_{\Gamma_k}^2.$$

Summing over all Γ_k^δ we get

$$\int_{\cup\Gamma_k^\delta} \phi_1^2 w^2 \ln|w| dx \le \varepsilon \int_{\cup\Gamma_k^\delta} \phi_1^2(|\nabla w|^2 + w^2)dx + \left(K - \frac{n+2A}{4}\ln\varepsilon\right)\|w\|_{\cup\Gamma_k}^2. \tag{2.31}$$

On the other hand on $\Omega \setminus \cup\Gamma_k^\delta$ we have that $\phi_1 \sim C$ and using the standard log Sobolev inequality we easily arrive at

$$\int_{(\cup\Gamma_k^\delta)^c} \phi_1^2 w^2 \ln|w| dx \le \varepsilon \int_{(\cup\Gamma_k^\delta)^c} \phi_1^2(|\nabla w|^2 + w^2)dx + \left(K - \frac{n}{4}\ln\varepsilon\right)\|w\|_{(\cup\Gamma_k)^c}^2, \tag{2.32}$$

when $n = 2$ (2.32) holds true for any $\nu > 2$ in place of n. Combining (2.31) and (2.32) we get that for $\|w\|_\Omega^2 = 1$, there holds

$$\int_\Omega \phi_1^2 w^2 \ln|w| dx \le \varepsilon \int_\Omega \phi_1^2(|\nabla w|^2 + w^2)dx + \left(K - \frac{n+2A}{4}\ln\varepsilon\right). \tag{2.33}$$

For a general $v \in C_0^\infty(\Omega)$ we apply (2.33) to $w = \frac{v}{\|v\|_\Omega}$ to get

$$\int_\Omega \phi_1^2 v^2 \ln\left(\frac{|v|}{\|v\|_\Omega}\right) dx \le \varepsilon \int_\Omega \phi_1^2(|\nabla v|^2 + v^2)dx + \left(K - \frac{n+2A}{4}\ln\varepsilon\right)\|v\|_\Omega^2. \tag{2.34}$$

Taking $u = \phi_1 v$, (2.25) follows.

$\square$

The above logarithmic Sobolev inequality is the main ingredient in establishing the intrinsic ultracontractivity of the semigroup generated by the operator $\mathcal{L}$ defined in (1.9). More precisely we have

Proposition 2.8 *Let $V \in L^1_{loc}(\Omega)$. We assume that (2.1) and (1.6) hold for some $\alpha_k > -\frac{k-2}{2} - \frac{n-k}{2(n-1)}$, $k = 1, \dots, n-1$, $\alpha_n \geq -\frac{n-2}{2}$. Then the operator $\mathcal{L}$ defined in (1.9) gives rise to an intrinsic ultracontractive semigroup in $L^2(\Omega)$, whose heat kernel $h(t,x,y)$ satisfies*

$$h(t,x,y) \leq C \max\{1, t^{-\frac{n+2A}{2}}\}\; d_1^{\alpha_1}(x) \cdots d_n^{\alpha_n}(x) d_1^{\alpha_1}(y) \cdots d_n^{\alpha_n}(y) e^{-\lambda_1 t}, \tag{2.35}$$

for any $t > 0$, $x, y \in \Omega$; here $A := max\{\alpha_1, \alpha_2, \dots, \alpha_n, 0\}$.

Proof: This is quite similar to Theorem 3.4 in [20] for this reason we only sketch it. We change variables by

$$v(x,t) := u(x,t)/\phi_1(x), \tag{2.36}$$

then if u solves problem (1.9) the function v satisfies

$$\begin{cases} \frac{\partial v}{\partial t} = -\mathcal{L}_{\phi_1} v := \frac{1}{\phi_1^2} \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(\phi_1^2 a_{ij}(x) \frac{\partial v}{\partial x_j}\right) - \lambda_1 v & \text{in} \quad (0,\infty) \times \Omega \ , \\ v(x,t) = 0 & \text{on} \quad (0,\infty) \times \partial_1 \Omega, \\ v(x,0) = v_0(x) & \text{on} \quad \Omega \ , \end{cases} \tag{2.37}$$

with $v_0(x) := u_0(x)\phi_1^{-1}(x)$.

We note that the elliptic operator $\mathcal{L}_{\phi_1} - \lambda_1$ is defined in the domain $D(\mathcal{L}_{\phi_1} - \lambda_1) := \{v \in H_0^1(\Omega; \phi_1^2) : \mathcal{L}_{\phi_1} - \lambda_1 \in L^2(\Omega, \phi_1^2(y)dy)\}$, where $H_0^1(\Omega; \phi_1^2)$ denotes the closure of $C_0^\infty(\Omega)$ functions with respect to the norm (2.5). To this elliptic operator it is naturally associated a bilinear symmetric form which is a Dirichlet form. Then Lemma 1.3.4 together with Theorems 1.3.2 and 1.3.3 in [9] imply that the elliptic operator $\mathcal{L}_{\phi_1} - \lambda_1$ generates an analytic semigroup, $e^{-(\mathcal{L}_{\phi_1} - \lambda_1)t}$, which is positivity preserving and contractive in $L^p(\Omega, \phi_1^2 dx)$ for any $1 \leq p \leq \infty$.

¿From the weighted logarithmic Sobolev inequality (2.34), we deduce the corresponding L^p logarithmic Sobolev inequality for any $p > 2$; to this end it is enough to apply (2.34) to the function $v := w^{\frac{p}{2}}$ for any smooth w. Using Theorem 2.2.7 in [9] - as it is used in Corollary 2.2.8 of [9] - we obtain that $e^{-(\mathcal{L}_{\phi_1} - \lambda_1)t}$ is an ultracontractive semigroup. As a consequence the semigroup $e^{-\mathcal{L}_{\phi_1} t}$ has a heat kernel h_{ϕ_1}, which satisfies the following uniform upper bound

$$h_{\phi_1}(t,x,y) \leq C \max\{1, t^{-\frac{n+2A}{2}}\} e^{-\lambda_1 t} \ \text{ for any } t > 0, x, y \in \Omega \ . \tag{2.38}$$

Clearly the heat kernel upper bound (2.35) associated to the operator $\mathcal{L}$ follows from (2.38), (1.6) and the fact that

$$h(t,x,y) = \phi_1(x)\phi_1(y) h_{\phi_1}(t,x,y) \ , \tag{2.39}$$

which is an immediate consequence of the change of variables (2.36). We omit further details.

□

3 Harnack inequalities and sharp heat kernel estimates

In this section we prove a parabolic Harnack inequality up to the boundary for the operator $\mathcal{L}_{\phi_1}$ defined in (2.37), and we deduce from it the corresponding heat kernel estimates as well as the proofs of Theorems 1.2 and 1.3 in the Introduction. We use Moser iteration technique, as adapted in [20] for bounded domains Ω. To this end we will prove four basic estimates. Namely, a sharp volume estimate, a local weighted Poincaré inequality, a local weighted Moser inequality and a density theorem.

We will use the following local representation of any smooth boundaryless hypersurface $\Gamma_{k,j}$ of codimension $k = 1, \ldots, n-1$, for any fixed $j = 1, \cdots, m_k$, which is of course Lipschitz. That is, we suppose there exists a finite number N (depending on both k and j) of coordinate systems (y_i, z_i), $y_i = (y_{i,1}, \cdots, y_{i,(n-k)})$ and $z_i = (z_{i,1}, \cdots, z_{i,k})$, for $i = 1, \cdots, N$, and the same number of functions $a_i = a_i(y_i) : I\!R^{n-k} \to I\!R^k$, ($a_i = (a_i^1, \ldots, a_i^k)$) defined on the closures of the $(n-k)$ dimensional cubes $\Delta_i := \{y_i : |y_{i,l}| \le \beta \text{ for } l = 1, \cdots, n-k\}$, $i \in \{1, \cdots, N\}$ so that for each point $x \in \Gamma_{k,j}$ there is at least one i such that $x = (y_i, a_i(y_i))$. The functions a_i satisfy the Lipschitz condition on $\overline{\Delta}_i$ with a constant $L > 0$ that is

$$|a_i(y_i) - a_i(\bar{y}_i)|_{I\!R^k} \le L|y_i - \bar{y}_i|_{I\!R^{n-k}}$$

for any $y_i, \bar{y}_i \in \overline{\Delta}_i$. We note $|y|_{I\!R^k}$ is the Euclidean norm in $I\!R^k$. Moreover, there exists a positive number $\beta < 1$, called the localization constant of $\Gamma_{k,j}$ and Ω, such that the set B_i defined for any $i \in \{1, \cdots, N\}$ by the relation

$$B_i = \{(y_i, z_i) : y_i \in \Delta_i, \;\; a_i^l(y_i) - \beta < z_{i,l} < a_i^l(y_i) + \beta\} \ ,$$

satisfies

$$U_i = B_i \cap \Omega = \{(y_i, z_i) : y_i \in \Delta_i, \;\; a_i^l(y_i) - \beta < z_{i,l} < a_i^l(y_i)\} \text{ if } k = 1 \ ,$$

or

$$U_i = B_i \cap \Omega = B_i = \{(y_i, z_i) : y_i \in \Delta_i, \;\; a_i^l(y_i) - \beta < z_{i,l} < a_i^l(y_i) + \beta\} \;\text{ if } k = 2, \ldots, n-1 \ ,$$

and $\Gamma_i = B_i \cap \partial\Omega = \{(y_i, z_i) : y_i \in \Delta_i, z_i = a_i(y_i)\}$. Finally, let us observe that for any $y \in U_i$ one has

$$(1 + L)^{-1}|a_i(y_i) - z_i|_{I\!R^k} \le d_k(y_i, z_i) \le |a_i(y_i) - z_i|_{I\!R^k} \ ;$$

see Corollary 4.8 in [25].

We fix a constant $\gamma \in (1, 2)$ and we define the "balls" we will use in Moser iteration technique. Roughly speaking they will be Euclidean balls if they stay away from the boundary and they will be n dimensional "deformed cubes", following the geometry of the boundary, if they are close enough to the boundary or if they intersect it. More precisely we have

Definition 3.1 *(i) For any $x \in \Omega$ and for any $0 < r < \beta$ we define the "ball" centered at x and having radius r as follows. $\mathcal{B}(x, r) = B(x, r)$ the Euclidean ball centered at x and having radius r if $d(x) \ge \gamma r$ (thus $d_k(x) \ge \gamma r$ for any $k = 1, \cdots, n$) or if $k = n$, while*

$$\mathcal{B}(x, r) = \{(y_i, z_i) : |y_i - x'|_{I\!R^{n-k}} < r,$$
$$a_i^l(y_i) - r - d_k(x) < z_{i,l} < a_i^l(y_i) + r - d_k(x) \text{ for any } l = 1, \cdots, k\}$$

if $k = 1, \ldots, n-1$ and $d_k(x) < \gamma r$ where $i \in \{1, \cdots, N\}$ is uniquely defined by the point $\bar{x} \in \Gamma_k$ such that $|\bar{x} - x|_{I\!R^n} = d_k(x)$, that is by the projection of the center x onto

$\Gamma_k \subset \partial\Omega$, and x' denotes the first $n-k$ coordinates of the point x in the i-orthonormal coordinate system. (ii) We also define the volume of the "ball" centered at x and having radius r by

$$V(x,r) := \int_{\mathcal{B}(x,r)\cap\Omega} \prod_{k=1}^{n} d_k^{2\alpha_k}(y)dy \ .$$

We first derive a sharp volume estimate, which plays a fundamental role in getting the sharp dependence of the heat kernel on x, y and t.

Lemma 3.2 (Sharp volume estimate) *Let $n \geq 2$ and $\alpha_k > -\frac{k}{2}$ for $k = 1, \dots, n$. Then, there exist positive constants c_1, c_2 and r_0 such that for any $x \in \Omega$ and $0 < r < r_0$, we have*

$$c_1 \prod_{k=1}^{n} (d_k(x)+r)^{2\alpha_k} r^n \leq V(x,r) \leq c_2 \prod_{k=1}^{n} (d_k(x)+r)^{2\alpha_k} r^n \ . \tag{3.1}$$

Proof of Lemma 3.2: Let us first consider the case where $d(x) \geq \gamma r$, whence $d_k(x) \geq \gamma r$ for any $k = 1, \dots, n$. Then $\mathcal{B}(x,r) = B(x,r) \subset \Omega$. Due to the fact that for any $y \in B(x,r)$ and any $k = 1, \cdots, n$ we have

$$\left(\frac{\gamma-1}{\gamma}\right) d_k(x) \leq d_k(x) - r \leq d_k(y) \leq d_k(x) + r \leq \left(\frac{\gamma+1}{\gamma}\right) d_k(x) \tag{3.2}$$

we easily get

$$V(x,r) \sim r^n \prod_{k=1}^{n} d_k^{2\alpha_k}(x) \sim r^n \prod_{k=1}^{n} (d_k(x)+r)^{2\alpha_k} \ ,$$

this proves the claim.

Let us now consider the case where $d(x) < \gamma r$. We claim that in this case there exists exactly one $k = 1, \cdots, n$ such that $d_k(x) < \gamma r$. This is due to the assumption that for some δ small enough $\Gamma^\delta_{k,j} \cap \Gamma^\delta_{l,i} = \emptyset$ for any $k \neq l$ and $i \neq j$ and $\Gamma^\delta_{k,j} \cap \Gamma^\delta_n = \emptyset$ for any $k = 1, \cdots, n-1$ and $j = 1, \cdots, m_k$, since we may suppose that $r < \frac{\delta}{2}$ (take $r_0 := \min\{\beta, \frac{\delta}{2}\}$, β being the localization constant of $\Gamma_{k,j}$ and Ω). Whence if $d_k(x) < \gamma r$ then $x \in \Gamma^\delta_k$ and $d_j(x) \geq \delta > \gamma r$ for any $j \neq k$, thus from (3.2) for any $y \in B(x,r)$ we have $d_j(y) \sim d_j(x)$; as a consequence

$$V(x,r) \sim \prod_{j=1, j\neq k}^{n} (d_j(x)+r)^{2\alpha_j} \int_{\mathcal{B}(x,r)\cap\Omega} d_k^{2\alpha_k}(y)dy \ ,$$

Hence the claim will follow as soon as we prove that

$$\int_{\mathcal{B}(x,r)\cap\Omega} d_k^{2\alpha_k}(y)dy \sim r^{n+2\alpha_k} \ . \tag{3.3}$$

Arguing as in (3.2) we have that $d_k(y) \leq (1+\gamma)r$ for any $y \in \mathcal{B}(x,r)$. Moreover one has that if $k \neq n$, $d_k(y) \geq r(\gamma-1)$ on a set of measure r^n. Indeed

$$\int_{\mathcal{B}(x,r)\cap\Omega\cap\{d_k(y)\geq r(\gamma-1)\}} dy = \int_{|y_i - x'|_{\mathbb{R}^{n-k}} \leq r} \int_{a_i^l(y_i)-r-d_k(x)}^{a_i^l(y_i)+r-r\gamma} dz_{i,l} dy_i =$$

$$= (2r - \gamma r + d_k(x))^k r^{n-k} \geq (2-\gamma)^k r^n,$$

and (3.3) follows. In the limit case $k = n$ see [29] for any $\alpha_n \in (-\frac{n}{2}, 0]$ (see also [28] and Lemma 2.3 in [20]). We note in fact that the same proof works for any $\alpha_n > -\frac{n}{2}$. □

¿From this one can easily deduce the doubling property:

Corollary 3.3 (Doubling property) *Let $n \geq 2$ and $\alpha_k > -\frac{k}{2}$ for $k = 1, \ldots, n$. Then, there exist positive constants C_D and r_0 such that for any $x \in \Omega$ and $0 < r < r_0$, we have*

$$V(x, 2r) \leq C_D V(x, r) \ .$$

Our next result reads:

Theorem 3.4 (Local weighted Poincaré inequality) *Let $n \geq 2$ and $\alpha_1 > 0$, $\alpha_k > -\frac{k}{2}$ for $k = 2, \cdots, n$. Then, there exist positive constants C_P and r_0 such that for any $x \in \Omega$ and $0 < r < r_0$, we have for all $f \in C^1(\overline{\mathcal{B}(x,r) \cap \Omega})$*

$$\inf_{\xi \in I\!R} \int_{\mathcal{B}(x,r)\cap\Omega} \prod_{k=1}^{n} d_k^{2\alpha_k}(y) |f(y) - \xi|^2 dy \leq C_P \ r^2 \int_{\mathcal{B}(x,r)\cap\Omega} \prod_{k=1}^{n} d_k^{2\alpha_k}(y) |\nabla f|^2 dy \ . \quad (3.4)$$

We note that our weight is not necessarily in the Muckenhoupt class A^2.

Proof: Let us first consider the case where $d(x) \geq \gamma r$. Then $\mathcal{B}(x, r) = B(x, r) \subset \Omega$ and for any $y \in B(x, r)$ and any $k = 1, \cdots, n$ we have $d_k(y) \sim d_k(x)$, as in estimate (3.2). Thus in this case (3.4) follows from the standard Poincaré inequality:

$$\inf_{\xi \in I\!R} \int_{B(x,r)} |f(y) - \xi|^2 dy \leq C_P \ r^2 \int_{B(x,r)} |\nabla f|^2 dy \ , \qquad f \in C^1(\overline{B(x,r)}) \ .$$

Let us now consider the case where $d_k(x) < \gamma r$, for some $k = 1, \cdots, n$. Then arguing as in Lemma 3.2 it is enough to prove the following for any $f \in C^1(\overline{\mathcal{B}(x,r) \cap \Omega})$ and any $k = 1, \cdots, n$

$$\inf_{\xi \in I\!R} \int_{\mathcal{B}(x,r)\cap\Omega} |f(y) - \xi|^2 d_k^{2\alpha_k}(y) dy \leq C_P \ r^2 \int_{\mathcal{B}(x,r)\cap\Omega} |\nabla f|^2 d_k^{2\alpha_k}(y) dy \ . \quad (3.5)$$

The case $k = 1$ corresponds to Theorem 2.5 in [20] (with $\lambda = 0$ there). The case $k = n$ has been treated in Theorem 3.1 in [29] (see also [28] and Theorem 2.5 in [20]) for any $\alpha_n \in (-\frac{n}{2}, 0]$. We note however that the same proof works for any $\alpha_n > -\frac{n}{2}$. So we need to consider the intermediate cases $k = 2, \ldots, n-1$.

We deduce (3.5) from the analogous statement for $k = n$, that is from the following inequality

$$\inf_{\xi \in I\!R} \int_{B(x,r)} |f(y) - \xi|^2 |y|_{I\!R^n}^{2\alpha_n} dy \leq C_P \ r^2 \int_{B(x,r)} |\nabla f|^2 |y|_{I\!R^n}^{2\alpha_n} dy, \qquad f \in C^1(\overline{B(x,r)}). \quad (3.6)$$

As a consequence of the local representation we have for some a and $s = (y, z)$

$$\int_{\mathcal{B}(x,r)\cap\Omega} d_k^{2\alpha_k}(s) |f(s) - \tilde{\xi}|^2 ds \leq$$

$$\le C(L)\int_{|y-x'|_{\mathbb{R}^{n-k}}\le r}\int_{a^l(y)-r-d_k(x)}^{a^l(y)+r-d_k(x)} |f(y,z)-\tilde{\xi}|^2|a(y)-z|_{\mathbb{R}^k}^{2\alpha_k}dz_l dy \le$$

$$\le C\int_{|y-x'|_{\mathbb{R}^{n-k}}\le r}\int_{d_k(x)-r}^{r+d_k(x)} |f(y,a(y)-w)-\tilde{\xi}|^2|w|_{\mathbb{R}^k}^{2\alpha_k}\ dw_l dy\ ,$$

here we used the following change of variables $(y,z)\to(y,w:=a(y)-z)$. Then, since

$$|f-\tilde{\xi}|^2\le 2\left(|f-\xi(y)|^2+|\xi(y)-\tilde{\xi}|^2\right)\ ,$$

where we use the following notation

$$\xi(y):=\frac{\int_{d_k(x)-r}^{r+d_k(x)} f(y,a(y)-w)|w|_{\mathbb{R}^k}^{2\alpha_k}dw}{\int_{d_k(x)-r}^{r+d_k(x)}|w|_{\mathbb{R}^k}^{2\alpha_k}dw}\ ,$$

$$\tilde{\xi}:=\omega_{n-k-1}^{-1}r^{-n+k}\int_{|y-x'|_{\mathbb{R}^{n-k}}\le r}\xi(y)dy\ ,$$

inequality (3.5) follows from estimates (i) and (ii) below.

(i) We have

$$\int_{|y-x'|_{\mathbb{R}^{n-k}}\le r}\left(\int_{d_k(x)-r}^{r+d_k(x)}|f(y,a(y)-w)-\xi(y)|^2|w|_{\mathbb{R}^k}^{2\alpha_k}\ dw_l\right)dy\le$$

$$\le C(r+d_k(x))^2\int_{|y-x'|_{\mathbb{R}^{n-k}}\le r}\left(\int_{d_k(x)-r}^{r+d_k(x)}|\nabla_z f|^2|w|_{\mathbb{R}^k}^{2\alpha_k}\ dw_l\right)dy\le$$

$$\le Cr^2\int_{\mathcal{B}(x,r)\cap\Omega}d_k^{2\alpha_k}(s)|\nabla f|^2ds\ ,$$

here we used the assumption $d_k(x)<\gamma r$ as well as inequality (3.6) applied in $\mathbb{R}^k$, instead of $\mathbb{R}^n$, this explains the restriction $2\alpha_k>-k$.

(ii) Finally

$$\int_{d_k(x)-r}^{r+d_k(x)}\left(\int_{|y-x'|_{\mathbb{R}^{n-k}}\le r}|\xi(y)-\tilde{\xi}|^2dy\right)|w|_{\mathbb{R}^k}^{2\alpha_k}\ dw_l\le$$

$$\le Cr^2\int_{d_k(x)-r}^{r+d_k(x)}\int_{|y-x'|_{\mathbb{R}^{n-k}}\le r}\left|\nabla_y f+\sum_{l=1}^k\frac{\partial f}{\partial z_l}\nabla_y a^l(y)\right|^2dy|w|_{\mathbb{R}^k}^{2\alpha_k}\ dw_l\le$$

$$\le Cr^2\int_{\mathcal{B}(x,r)\cap\Omega}d_k^{2\alpha_k}|\nabla f|^2dydz\ ,$$

here we used the standard Poincaré inequality on the Euclidean $n-k$ dimensional ball of radius r centered at x'.

The proof of inequality (3.5) is now complete.

$\square$

All the ingredients of the abstract machinery of [24] are now in place. However, since bounded domains endowed with the Euclidean metric are not complete manifolds, the standard method should be modified as in [20]. In particular we will next prove a local weighted Moser inequality as well as a density Theorem which are crucial in making the Moser iteration to work in our setting.

We next prove the following local weighted Moser inequality:

Theorem 3.5 (Local weighted Moser inequality) *Let $n \geq 2$ and $\alpha_1 > 0$, $\alpha_k \geq -\frac{k-2}{2}$ for $k = 2, \cdots, n$. Then, there exist positive constants C_M and r_0 such that for any $\nu \geq n + 2A$, $A := max\{\alpha_1, \alpha_2, \dots, \alpha_n, 0\}$, $x \in \Omega$, $0 < r < r_0$ and $f \in C_0^\infty(\mathcal{B}(x,r) \cap \Omega)$ we have*

$$\int_{\mathcal{B}(x,r)\cap\Omega} |f(y)|^{2\left(1+\frac{2}{\nu}\right)} \prod_{i=1}^n d_i^{2\alpha_i}(y)dy \leq$$

$$\leq C_M r^2 V(x,r)^{-\frac{2}{\nu}} \left(\int_{\mathcal{B}(x,r)\cap\Omega} |\nabla f|^2 \prod_{i=1}^n d_i^{2\alpha_i}(y)dy \right) \left(\int_{\mathcal{B}(x,r)\cap\Omega} f^2 \prod_{i=1}^n d_i^{2\alpha_i}(y)dy \right)^{\frac{2}{\nu}}. \tag{3.7}$$

Proof: Let us first consider the case where $d(x) \geq \gamma r$. Then $\mathcal{B}(x,r) = B(x,r) \subset \Omega$ and for any $y \in B(x,r)$ and any $k = 1, \cdots, n$ we have $d_k(y) \sim d_k(x)$, as in estimate (3.2), and the claim follows from the standard Moser inequality, which we recall here: There exists a positive constant C such that for any $x \in \Omega$, $r > 0$, and any $\nu \geq n$ if $n \geq 3$ or any $\nu > 2$ if $n = 2$, the following holds true

$$\int_{B(x,r)} |f(y)|^{2\left(1+\frac{2}{\nu}\right)} dy \leq C r^2 r^{-\frac{2n}{\nu}} \left(\int_{B(x,r)} |\nabla f|^2 dy \right) \left(\int_{B(x,r)} f^2 dy \right)^{\frac{2}{\nu}} ,$$

for all $f \in C_0^\infty(B(x,r))$ (see for example Section 2.1.3 in [30]). Making use of the sharp volume estimate in Lemma 3.2, we have

$$\int_{B(x,r)} \prod_{i=1}^n d_i^{2\alpha_i}(y) |f(y)|^{2\left(1+\frac{2}{\nu}\right)} dy \leq C \prod_{i=1}^n d_i^{2\alpha_i}(x) r^2 r^{-\frac{2n}{\nu}} \left(\int_{B(x,r)} |\nabla f|^2 dy \right) \left(\int_{B(x,r)} f^2 dy \right)^{\frac{2}{\nu}} \leq$$

$$\leq C \left(\prod_{i=1}^n d_i^{2\alpha_i}(x) \right)^{1-1-\frac{2}{\nu}} r^2 r^{-\frac{2n}{\nu}} \left(\int_{B(x,r)} \prod_{i=1}^n d_i^{2\alpha_i}(y) |\nabla f|^2 dy \right) \left(\int_{B(x,r)} \prod_{i=1}^n d_i^{2\alpha_i}(y) f^2 dy \right)^{\frac{2}{\nu}} =$$

$$= C_M r^2 V(x,r)^{-\frac{2}{\nu}} \left(\int_{B(x,r)} \prod_{i=1}^n d_i^{2\alpha_i}(y) |\nabla f|^2 dy \right) \left(\int_{B(x,r)} \prod_{i=1}^n d_i^{2\alpha_i}(y) f^2 dy \right)^{\frac{2}{\nu}} ,$$

and (3.7) has been proved in case $d(x) \geq \gamma r$.

Let us now consider the case where $d(x) < \gamma r$. Arguing as in Lemma 3.2 this corresponds to consider the case where $d_k(x) < \gamma r$ for some $k = 1, \dots, n$. In view of (3.1) it is enough to prove

$$\int_{\mathcal{B}(x,r)\cap\Omega} |f(y)|^{2\left(1+\frac{2}{\nu}\right)} d_k^{2\alpha_k}(y)dy \leq$$

$$\leq C_M r^2 r^{-\frac{2(n+2\alpha_k)}{\nu}} \left(\int_{\mathcal{B}(x,r)\cap\Omega} |\nabla f|^2 d_k^{2\alpha_k}(y)dy \right) \left(\int_{\mathcal{B}(x,r)\cap\Omega} f^2 d_k^{2\alpha_k}(y)dy \right)^{\frac{2}{\nu}}. \tag{3.8}$$

In the argument that follows we omit the integral set which is always taken as $\mathcal{B}(x,r) \cap \Omega$ and we define $d\mu := d_k^{2\alpha_k}(y)dy$. First of all making use twice of Hölder inequality for any $\nu > n + 2\alpha_k^+$, we have

$$\int f^{2\left(1+\frac{2}{\nu}\right)} d\mu \leq \left(\int f^{2\left(1+\frac{2}{n+2\alpha_k^+}\right)} d\mu \right)^{\frac{\nu+2}{\nu} \frac{n+2\alpha_k^+}{n+2\alpha_k^+ + 2}} \left(\int d\mu \right)^{1-\frac{\nu+2}{\nu} \frac{n+2\alpha_k^+}{n+2\alpha_k^+ + 2}} , \tag{3.9}$$

as well as

$$\left(\int f^2 d\mu\right)^{\frac{2(\nu-n-2\alpha_k^+)}{\nu(n+2\alpha_k^+)}} \le \left(\int f^{2\left(1+\frac{2}{n+2\alpha_k^+}\right)} d\mu\right)^{\frac{2(\nu-n-2\alpha_k^+)}{\nu(n+2\alpha_k^++2)}} \left(\int d\mu\right)^{\frac{4(\nu-n-2\alpha_k^+)}{\nu(n+2\alpha_k^+)(n+2\alpha_k^++2)}} , \tag{3.10}$$

multiplying both sides of inequalities (3.9) and (3.10) we deduce that

$$\int f^{2\left(1+\frac{2}{\nu}\right)} d\mu \le \left(\int f^{2\left(1+\frac{2}{n+2\alpha_k^+}\right)} d\mu\right) \left(\int f^2 d\mu\right)^{\frac{2(n+2\alpha_k^+-\nu)}{\nu(n+2\alpha_k^+)}} \left(\int d\mu\right)^{\frac{2(\nu-n-2\alpha_k^+)}{\nu(n+2\alpha_k^+)}} . \tag{3.11}$$

Hölder inequality also implies that

$$\int f^{2\left(1+\frac{2}{n+2\alpha_k^+}\right)} d\mu \le \left(\int f^{\frac{2(n+2\alpha_k^+)}{n+2\alpha_k^+-2}} d\mu\right)^{\frac{n+2\alpha_k^+-2}{n+2\alpha_k^+}} \left(\int f^2 d\mu\right)^{\frac{2}{n+2\alpha_k^+}} . \tag{3.12}$$

To continue we will use the following local weighted Sobolev inequality

$$\left(\int_{\mathcal{B}(x,r)\cap\Omega} d_k^{2\alpha_k}(y)|f(y)|^{\frac{2(n+2\alpha_k^+)}{n+2\alpha_k^+-2}} dy\right)^{\frac{n+2\alpha_k^+-2}{n+2\alpha_k^+}} \le C_S \int_{\mathcal{B}(x,r)\cap\Omega} d_k^{2\alpha_k}(y)|\nabla f|^2 dy . \tag{3.13}$$

Then from (3.11), (3.12), (3.13) and (3.3) we get the desired Moser inequality (3.8) with

$$C_M = C_S \, r^{-\frac{4\alpha_k^-}{n+2\alpha_k^+}} ,$$

where $\alpha_k^- := \max\{0, -\alpha_k\}$. It remains to prove (3.13) with a positive constant C_S independent of x and r for $0 < r < r_0$. This is a consequence of (2.28) and the local weighted Poincaré inequality (3.4) (since for functions $f \in C_0^\infty(\mathcal{B}(x,r)\cap\Omega)$ one can take $\xi = 0$ in it). It follows that C_M is automatically independent of r if $\alpha_k \ge 0$. Hence Theorem 3.5 has been proved in this case.

It remains to show that C_M is independent of r also in the case $\alpha_k < 0$. In fact in such a case instead of (3.13) we have an even better estimate (see the definition of β_k given in (2.9) for $q = \frac{2n}{n-2}$ and $n \ge 3$)

$$\left(\int_{\mathcal{B}(x,r)\cap\Omega} d_k^{\frac{2n\alpha_k}{n-2}}(y)|f(y)|^{\frac{2n}{n-2}} dy\right)^{\frac{n-2}{n}} \le \tilde{C}_S \int_{\mathcal{B}(x,r)\cap\Omega} d_k^{2\alpha_k}(y)|\nabla f|^2 dy ,$$

for some positive constant r_0 and for any $x \in \Omega$, $0 < r < r_0$, $f \in C_0^\infty(\mathcal{B}(x,r)\cap\Omega)$, with a positive constant $\tilde{C}_S$ independent from x and r. Whence

$$\left(\int_{\mathcal{B}(x,r)\cap\Omega} d_k^{2\alpha_k}(y)|f(y)|^{\frac{2n}{n-2}} dy\right)^{\frac{n-2}{n}} \le \tilde{C}_S \left(\max_{y\in\mathcal{B}(x,r)\cap\Omega} d_k(y)\right)^{\frac{4\alpha_k^-}{n}} \int_{\mathcal{B}(x,r)\cap\Omega} d_k^{2\alpha_k}(y)|\nabla f|^2 dy \le$$

$$\leq \tilde{C}_S \, r^{\frac{4\alpha_k^-}{n}} \int_{\mathcal{B}(x,r)\cap\Omega} d_k^{2\alpha_k}(y)|\nabla f|^2 dy \ ,$$

since $d_k(y) \leq d_k(x) + r \leq (1+\gamma)r$. If we use the above inequality in the place of (3.13) we get $C_M = \tilde{C}_S$, that is C_M is independent of r.
The proof of Theorem 3.5 is now complete.

□

Finally let us prove the following density Theorem, which as explained in [20] is crucial for the Moser iteration to work on bounded domains. Let

$$H^1(\Omega;\, \phi_1^2) = \{v \in L^2(\Omega, \phi_1^2 dx) : \int_\Omega \phi_1^2 |\nabla v|^2 dx < +\infty\},$$

with norm defined by $||v||^2_{H^1_{\phi_1}} := \int_\Omega \phi_1^2(v^2 + |\nabla v|^2)dx$.

Theorem 3.6 (Density theorem) *Let $n \geq 2$. Suppose that ϕ_1 satisfies*

$$c_1 d_1^{\alpha_1}(x) \ldots d_n^{\alpha_n}(x) \leq \phi_1(x) \leq c_2 d_1^{\alpha_1}(x) \ldots d_n^{\alpha_n}(x),$$

for $x \in \Omega$ with c_1, c_2 positive constants and $\alpha_k \geq -\frac{k-2}{2}$, $k = 1, \ldots, n$. Then, the

$$C_0^\infty(\Omega) \ functions \ are \ dense \ in \ H^1(\Omega;\, \phi_1^2).$$

Proof: The special case $\alpha_k = 0$ for $k = 2, \ldots, n$ was treated in Theorem 2.11 of [20]. First of all from Theorem 7.2 in [25] it is known that the set $C^\infty(\overline{\Omega})$ is dense in $H^1(\Omega; \phi_1^2)$. Thus for any $v \in H^1(\Omega; \phi_1^2)$ there exists $v_m \in C^\infty(\overline{\Omega})$ such that for any $\epsilon > 0$ we have $||v - v_m||_{H^1_{\phi_1}} \leq \epsilon$ if $m \geq m(\epsilon)$. Let us choose $w := v_{m(\epsilon)}$ and let us define, for any $k = 1, \cdots, n$ and any $j \geq 1$, the following function

$$\psi_k^j(x) = \begin{cases} 0 & \text{if } \ d_k(x) \leq \frac{1}{j^2} \ , \\ 1 + \frac{\ln(j d_k(x))}{\ln(j)} & \text{if } \ \frac{1}{j^2} < d_k(x) < \frac{1}{j} \ , \\ 1 & \text{if } \ d_k(x) \geq \frac{1}{j} \ . \end{cases}$$

Then $w^j := w \prod_{k=1}^n \psi_k^j \in C_0^{0,1}(\Omega)$, and

$$||w - w^j||_{H^1_{\phi_1}} = ||w(1 - \prod_{k=1}^n \psi_k^j)||_{H^1_{\phi_1}} \leq 2 \int_\Omega (w^2 + |\nabla w|^2)(1 - \prod_{k=1}^n \psi_k^j)^2 \phi_1^2(y) \ dy +$$

$$+2 \int_\Omega w^2 \sum_{k=1}^n |\nabla \psi_k^j|^2 \phi_1^2(y) \ dy \leq$$

$$\leq 2 \int_{\cup_{k=1}^n \{d_k(y) < \frac{1}{j}\}} (w^2 + |\nabla w|^2)\phi_1^2 dy + 2 \sum_{k=1}^n \int_{\frac{1}{j^2} < d_k(y) < \frac{1}{j}} \frac{w^2 |\nabla d_k|^2}{d_k^2(y)(\ln(j))^2} d_k^{2\alpha_k}(y) dy \ .$$

Now as $j \to \infty$ it is clear that the first term in the right hand side goes to zero since $w \in H^1_{\phi_1}$. We next show that also the second term goes to zero. Recalling that $|\nabla d_k| \leq 1$

$$\int_{\frac{1}{j^2} < d_k(y) < \frac{1}{j}} \frac{w^2 |\nabla d_k|}{d_k^2(y)(\ln(j))^2} d_k^{2\alpha_k}(y) dy \leq C \frac{||w||^2_{L^\infty(\Omega)}}{(\ln(j))^2} \int_{\frac{1}{j^2} < t < \frac{1}{j}} t^{2\alpha_k - 2} t^{k-1} dt \leq$$

$$\leq C \frac{||w||^2_{L^\infty(\Omega)}}{(\ln(j))^2} \frac{1}{j^{2\alpha_k - 2 + k}} \int_{\frac{1}{j^2} < t < \frac{1}{j}} t^{-1} dt \leq C \frac{||w||^2_{L^\infty(\Omega)}}{\ln(j)} \to 0 \ ,$$

as $j \to \infty$ for any $2\alpha_k - 2 + k \geq 0$, and this completes the proof.

□

At this point we have all the ingredients needed in order to apply Moser iteration technique up to the boundary, as adapted on bounded domains in [20], to the operator

$$\mathcal{L}_\alpha v := -\frac{1}{\prod_{k=1}^n d_k^{2\alpha_k}(x)} \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}(x) \ \prod_{k=1}^n d_k^{2\alpha_k}(x) \frac{\partial v}{\partial x_j} \right) + \lambda_1 v \ , \tag{3.14}$$

or equivalently to the degenerate elliptic operator $\mathcal{L}_{\phi_1}$, defined in (2.37).

In fact one can prove the following result

Theorem 3.7 (Parabolic Harnack inequality up to the boundary) *Let $n \geq 2$, $\Omega \subset I\!R^n$, be a smooth bounded domain, $\lambda_1 \in I\!R$ and $\alpha_k \geq -\frac{k-2}{2}$, for $k = 1, \cdots, n$. Then there exist positive constants C_H and $R = R(\Omega)$ such that for $x \in \Omega$, $0 < r < R$ and for any positive solution $v(y,t)$ of*

$$\frac{\partial v}{\partial t} = -\mathcal{L}_\alpha v \ \textit{in} \ \{\mathcal{B}(x,r) \cap \Omega\} \times (0, r^2) \ , \tag{3.15}$$

the following estimate holds true

$$\text{ess sup}_{(y,t) \in \left\{\mathcal{B}(x,\frac{r}{2}) \cap \Omega\right\} \times (\frac{r^2}{4}, \frac{r^2}{2})} v(y,t) \leq C_H \ \text{ess inf}_{(y,t) \in \left\{\mathcal{B}(x,\frac{r}{2}) \cap \Omega\right\} \times (\frac{3}{4} r^2, r^2)} v(y,t) \ .$$

Here we use the following definition of solutions:

Definition 3.8 *By a solution $v(y,t)$ to (3.15), we mean a function*

$$v \in C^1 \left((0, r^2); L^2 \left(\mathcal{B}(x,r) \cap \Omega, \prod_{k=1}^n d_k^{2\alpha_k}(y) dy \right) \right) \cap C^0 \left((0, r^2); H^1 \left(\mathcal{B}(x,r) \cap \Omega, \prod_{k=1}^n d_k^{2\alpha_k}(y) dy \right) \right)$$

such that for any $\Phi \in C^0((0,r^2); C_0^\infty(\mathcal{B}(x,r) \cap \Omega))$ and any $0 < t_1 < t_2 < r^2$ we have

$$\int_{t_1}^{t_2} \int_{\mathcal{B}(x,r) \cap \Omega} \left\{ v_t \Phi + \sum_{i,j=1}^n a_{ij}(y) \frac{\partial v}{\partial x_i} \frac{\partial \Phi}{\partial x_j} + \lambda_1 v \ \Phi \right\} \prod_{k=1}^n d_k^{2\alpha_k}(y) dy dt = 0 \ . \tag{3.16}$$

Let us note that Theorem 3.7 is sharp, in the sense that the same statement does not hold true if $\alpha_k < -\frac{k-2}{2}$ for some $k = 1, \ldots, n$ as explained in [20].

The parabolic Harnack inequality up to the boundary for the Schrödinger type operator $\mathcal{L}$ defined in (1.9) and stated in Theorem 1.2 is proved as follows:

Proof of Theorem 1.2: Clearly Theorem 3.7 applies also to the operator $\mathcal{L}_{\phi_1}$ instead of $\mathcal{L}_\alpha$. Hence Theorem 1.2 is a consequence of Theorem 3.7 for $\mathcal{L}_{\phi_1}$ and of the change of variables $v = u\phi_1^{-1}$ see (2.36).

□

¿From the parabolic Harnack inequality in Theorem 3.7 we deduce, as in [20], the sharp two-sided estimates for the heat kernel l_α associated to the elliptic operator $\mathcal{L}_\alpha$ defined in (3.14) under Dirichlet boundary conditions. That is

$$v(x,t) := \int_\Omega l_\alpha(t,x,y) v_0(y) \prod_{i=1}^n d_i^{2\alpha_i}(y) dy$$

satisfies $v_t = -\mathcal{L}_\alpha v$ in $(0,\infty)\times\Omega$, $v(0,x) = v_0(x)$ on Ω and $v = 0$ on $(0,\infty)\times\partial_1\Omega$. We then have

Theorem 3.9 *Let $n \geq 2$, $\Omega \subset I\!R^n$, be a smooth bounded domain, $\lambda_1 \in I\!R$ and $\alpha_k \geq -\frac{k-2}{2}$, for $k = 1,\cdots,n$. Then there exist positive constants C_1, C_2, with $C_1 \leq C_2$, and $T > 0$ depending on Ω such that*

$$C_1 \prod_{i=1}^n (d_i(x)+\sqrt{t})^{-\alpha_i}(d_i(y)+\sqrt{t})^{-\alpha_i} t^{-\frac{n}{2}} e^{-C_2\frac{|x-y|^2}{t}} \leq l_\alpha(t,x,y) \leq$$

$$\leq C_2 \prod_{i=1}^n (d_i(x)+\sqrt{t})^{-\alpha_i}(d_i(y)+\sqrt{t})^{-\alpha_i} t^{-\frac{n}{2}} e^{-C_1\frac{|x-y|^2}{t}}$$

for all $x, y \in \Omega$ and $0 < t \leq T$.

Finally from the global upper bound in (2.38), arguing as in Theorem 6 of [7] (see also Proposition 4 in [8] as well as the proof of Theorem 1.2 in [20]), one can deduce an analogous lower bound for large times, thus obtaining the following sharp long-time asymptotics of the heat kernel

Theorem 3.10 *Let $n \geq 2$, $\Omega \subset I\!R^n$, be a smooth bounded domain, $\lambda_1 \in I\!R$ and $\alpha_k \geq -\frac{k-2}{2}$, for $k = 1,\cdots,n$. Then there exist positive constants C_1, C_2, with $C_1 \leq C_2$, and $T > 0$ depending on Ω such that*

$$C_1 e^{-\lambda_1 t} \leq l_\alpha(t,x,y) \leq C_2 e^{-\lambda_1 t}$$

for all $x, y \in \Omega$ and $t \geq T$.

¿From Theorems 3.9 and 3.10, making use of the equivalence (2.39) as well as of assumption (1.6), we get the corresponding result for the Schrödinger operator $\mathcal{L}$ stated in Theorems 1.3 in the Introduction. We omit further details.

As we have already mentioned integrating the sharp two-sided estimates for $h(t,x,y)$ in Theorem 1.3 with respect to the time variable, one can deduced estimates on the Green function for the Schrödinger operator $\mathcal{L}$ defined in (1.9) in the case $\lambda_1 > 0$. Some explicit examples of sharp two sided Green function estimates are given in Theorem 4.11 in [20].

4 Applications

In this Section we give some examples of singular potentials V for which the results of the present work apply; that is, we give examples of potentials V for which the generalized first eigenvalue is not $-\infty$, and the corresponding first eigenfunction is bounded from above and below uniformly by some power of the distance function. We should stress that the asymptotics of ϕ_1 for the examples that follow is a consequence only of the maximum principle as used in [5]. We will present the detailed argument for example III in the Appendix; the other cases can be treated similarly.

To this end let us first prove that the sum of two potentials having disjoint singularity sets and finite generalized first eigenvalues, also has finite generalized first eigenvalue.

Lemma 4.1 *Let V_i, $i = 1,2$ be such that $V_i \in L^1_{loc}(\Omega) \cap L^\infty_{loc}(\overline{\Omega} \setminus S_i)$, where S_i are compact subsets of $\overline{\Omega}$ such that $S_1 \cap S_2 = \emptyset$ and*

$$\lambda_1(V_i) := \inf_{u \in C_0^\infty(\Omega)} \frac{\int_\Omega \left(|\nabla u|^2 - V_i u^2\right) dx}{\int_\Omega u^2 dx} .$$

Assuming that $\lambda_1(V_i) > -\infty$ for $i = 1,2$ then $\lambda_1(V_1 + V_2) > -\infty$.

Proof: Let us take $\varphi \in C^\infty(\overline{\Omega})$, $0 \le \varphi \le 1$, such that $\varphi = 1$ in $\overline{\Omega} \cap \overline{\Omega}_1$ and $\varphi = 0$ in $\overline{\Omega} \setminus \tilde{\Omega}_1$ where Ω_1 is a neighborhood of S_1, that is $\Omega_1 := \{x \in \overline{\Omega} : \mathrm{dist}(x, S_1) < \delta\}$ for some small $\delta > 0$, and $\tilde{\Omega}_1$ is a slightly bigger neighborhood of S_1, thus $S_1 \subset \Omega_1 \subset \tilde{\Omega}_1$. Whence $\Omega_2 := \overline{\Omega} \setminus \tilde{\Omega}_1$ is a neighborhood of S_2 and $\tilde{\Omega}_2 := \overline{\Omega} \setminus \Omega_1$ is a slightly bigger neighborhood of S_2, thus $\Omega_2 \subset \tilde{\Omega}_2$. Whence for any $u \in C_0^\infty(\Omega)$ we have $u = u\varphi + u(1-\varphi) =: u_1 + u_2$. By elementary calculations we have that

$$\int_\Omega |\nabla u|^2 dx = \int_\Omega |\nabla(u\varphi + (1-\varphi)u)|^2 dx \ge \int_\Omega |\nabla u_1|^2 dx + \int_\Omega |\nabla u_2|^2 dx - K \int_\Omega u^2 dx$$

for a suitable positive constant K. Then for $V := V_1 + V_2$ we have

$$\int_\Omega \Big(|\nabla u|^2 - V u^2\Big)\, dx \ge$$

$$\ge \int_\Omega \Big(|\nabla u_1|^2 - V_1 u_1^2 + |\nabla u_2|^2 - V_2 u_2^2 + u^2 (V_1 \varphi^2 + V_2 (1-\varphi)^2 - V - K)\Big)\, dx =$$

$$= \int_{\tilde{\Omega}_1} \Big(|\nabla u_1|^2 - V_1 u_1^2\Big)\, dx + \int_{\tilde{\Omega}_2} \Big(|\nabla u_2|^2 - V_2 u_2^2\Big)\, dx +$$

$$+ \int_\Omega u^2 \Big[V_1(\varphi^2 - 1) + V_2((1-\varphi)^2 - 1) - K\Big]\, dx \ge$$

$$\ge \Big(\lambda_1(V_1) + \lambda_1(V_2) - ||V_1||_{L^\infty(\tilde{\Omega}_2)} - ||V_2||_{L^\infty(\tilde{\Omega}_1)} - K\Big) \int_\Omega u^2 dx .$$

□

We now present some concrete examples.

Example I Our first example is motivated by [15], [16], [17] and deals with multipolar inverse-square potentials. Let $n \ge 3$, $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain from which we have removed m points $x_1, \ldots, x_m$ and

$$V(x) = \sum_{i=1}^m \frac{c_i}{|x - x_i|^2} ,$$

for $0 \le c_i \le \frac{(n-2)^2}{4}$. We note that differently from [15], we may take in each one of the inverse-square potentials the critical Hardy constant. This is due to the fact that we study the Schrödinger operator $-\Delta - V$ on a bounded domain. In such a case one can prove that $\phi_1(x) \sim \prod_{i=1}^m |x - x_i|^{\beta_i} \mathrm{dist}(x, \partial_1\Omega)$ with

$$\beta_i := \frac{2 - n + \sqrt{(n-2)^2 - 4c_i}}{2} ,$$

see Lemma 7 in [13] and Theorem 7.1 in [11] on one hand and the elliptic regularity on the other. In fact the function $f(x) := |x - x_i|^{\beta_i}$ satisfies the equation $\Delta f + \frac{c_i}{|x-x_i|^2} f = 0$ in $I\!\!R^n \setminus \{x_i\}$. We only need to check that $\lambda_1 > -\infty$. This follows from Lemma 4.1, which clearly can be generalized to a finite sum of potentials, and the improved L^2 inequality given in [31] for a single inverse-square potential. Consequently Theorems 1.2, 1.3 and 2.1 apply with $\Gamma_k = \emptyset$ for $k = 2, \ldots, n-1$ and $\Gamma_n = \{x_1, \ldots, x_m\}$. Note that in this case $d_n^{\alpha_n}(x)$ stands for $\prod_{i=1}^m |x - x_i|^{\beta_i}$, that is, we have m different sets of the same codimension n in the boundary of Ω, where ϕ_1 may present different degeneracies.

Example II Let $n \geq 4$, $\Omega = B_R \setminus E$, for some $R > 1$, where

$$E := \{x \in I\!\!R^n : x_1^2 + x_2^2 = 1,\ x_3 = \ldots = x_n = 0\}$$

$$B_R := \{x \in I\!\!R^n : x_1^2 + \ldots + x_n^2 < R^2\} \quad \text{and}$$

$$V(x) = \frac{1}{4} \frac{1}{\mathrm{dist}^2(x, \partial B_R)} + \frac{(n-3)^2}{4} \frac{1}{\mathrm{dist}^2(x, E)} .$$

In such a case one can easily prove that $\phi_1(x) \sim \mathrm{dist}^{\frac{1}{2}}(x, \partial B_R)\ \mathrm{dist}^{\frac{3-n}{2}}(x, E)$, see [13]. The fact that $\lambda_1 > -\infty$ follows making use of Lemma 4.1, from the improved L^2 inequality given in [4] for the inverse-square distance to ∂B_R and the one given in [13] or [19] for the inverse-square distance to the set E having codimension $n-1$. Theorems 1.2, 1.3 and 2.1 now apply to $-\Delta - V$ with $\alpha_1 = 1/2$ and $\alpha_{n-1} = (3-n)/2$, whereas all the other α_k's are zero.

Example III Let $n \geq 2$, and $\Omega \subset I\!\!R^n$ be a bounded domain such that $\partial\Omega = \partial_1\Omega$, that is, the boundary of Ω has codimension one. We now take

$$V(x) = \frac{1}{4} \frac{1}{\mathrm{dist}^2(x, \partial\Omega)} .$$

By the results of [4] we have that $\lambda_1 > -\infty$ under appropriate regularity assumptions on $\partial\Omega$. We recall also that $\phi_1(x) \sim \mathrm{dist}^{\frac{1}{2}}(x, \partial\Omega)$, as shown in [13]; see also Appendix, where we will provide a self-contained proof based only on the maximum principle. Therefore Theorems 1.2, 1.3 and 2.1 apply to $-\Delta - V$ with $\alpha_1 = 1/2$, whereas all the other α_k's are zero. We note that this improves the corresponding Theorems in [FMT1] removing the convexity assumption, under which it is known that $\lambda_1 > 0$ (see [4]).

Example IV Let $n \geq 3$, $\Omega' \subset I\!\!R^n$ be a smooth bounded domain containing the origin, $\Omega = \Omega' \setminus \{0\}$, and

$$V(x) = \frac{1}{4} \frac{1}{\mathrm{dist}^2(x, \partial_1\Omega)} + \frac{(n-2)^2}{4} \frac{1}{|x|^2} .$$

The fact that $\lambda_1 > -\infty$ may be deduced from Lemma 4.1 making use of the L^2 improved Hardy inequality in [31] for the inverse-square potential $1/|x|^2$ and of the one in [4] for the inverse-square potential involving the distance to $\partial_1\Omega$. In this example we have $\phi_1(x) \sim \mathrm{dist}^{\frac{1}{2}}(x, \partial_1\Omega)|x|^{\frac{2-n}{2}}$. Whence Theorems 1.2, 1.3 and 2.1 apply to $-\Delta - V$ with $\alpha_1 = 1/2$ and $\alpha_n = (2-n)/2$, whereas all the other α_k's are zero.

Example V Let $n \geq 3$, and $B_1 \subset I\!R^n$ be the unit ball. For $-\frac{n-2}{2} \leq a < 0$ we consider the operator $\mathcal{L} = -L - V$ where

$$Lu = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u}{\partial x_j} \right), \qquad a_{ij} = \delta_{ij} + \frac{1}{2}|x|^{2-a}(1 - \delta_{ij}),$$

and $V(x) = -\frac{a(n+a-2)}{|x|^2}$. The operator L is easily seen to be uniformly elliptic and in fact

$$\left(1 + \frac{n}{2}\right) |\xi|^2 \geq \sum_{i,j=1}^{n} a_{ij} \xi_i \xi_j \geq \frac{1}{2} |\xi|^2. \tag{4.1}$$

On the other hand if

$$Q[u] = \int_{B_1} \left(\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} - V u^2 \right) dx,$$

using the change of variables $u = |x|^a v$ a straightforward calculation shows that

$$\int_{B_1} \left(\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} - V u^2 \right) dx = \int_{B_1} (\delta_{ij}|x|^{2a} + \frac{1}{2}|x|^{a+2}(1 - \delta_{ij})) v_{x_i} v_{x_j} dx.$$

Using (4.1) one can easily see that $\lambda_1 > 0$.

In this example we have that $\phi_1(x) \sim \mathrm{dist}(x, \partial B_1)|x|^a$. We can apply our results with $\alpha_1 = 1$ and $\alpha_n = a$, whereas all the other α_k's are zero.

5 Appendix

In this Appendix we consider the operator $\mathcal{L} := -\Delta - \frac{1}{4}\frac{1}{d^2(x)}$ which corresponds to Example III of Section 4 and we will prove that the corresponding eigenfunction ϕ_1 is such that $\phi_1(x) \sim d^{\frac{1}{2}}(x)$, where $d(x) := \mathrm{dist}(x, \partial\Omega)$.

Step I: Existence of ϕ_1 in a suitable energy space. Let $\eta \in C^2(\Omega)$ be a function such that $\eta(x) = d^{1/2}(x)$ near the boundary, say, $d(x) \leq \epsilon_0$, and $\eta(x) \geq c_0 > 0$ for $d(x) \geq \epsilon_0$. Let $H^1_d(\Omega)$ be the closure of $C_0^\infty(\Omega)$ functions under the norm

$$||v||^2_{H^1_d} := \int_\Omega d \left(|\nabla v|^2 + v^2 \right) dx \ .$$

This norm is equivalent to the norm $\|v\| := \int_\Omega \eta^2 \left(|\nabla v|^2 + v^2 \right) dx$. Changing variables by $u = \eta v$, in

$$-\infty < \lambda_1 = \inf_{u \in C_0^\infty(\Omega)} \frac{\int_\Omega (|\nabla u|^2 - \frac{u^2}{4d^2}) dx}{\int_\Omega u^2 dx},$$

we get the equivalent inequality

$$-\infty < \lambda_1 = \inf_{v \in C_0^\infty(\Omega)} \frac{\int_\Omega (\eta^2 |\nabla v|^2 - (\eta \Delta \eta + \frac{\eta^2}{4d^2}) v^2) dx}{\int_\Omega \eta^2 v^2 dx}. \tag{5.1}$$

Using the fact that $\eta\Delta\eta + \frac{\eta^2}{4d^2} \in L^\infty(\Omega)$ as well as the following estimate

$$\int_\Omega \frac{X^2(d)}{d} v^2 dx \leq C \int_\Omega d(|\nabla v|^2 + v^2) dx \ , \qquad v \in C_0^\infty(\Omega),$$

which was established in Proposition 5.1 in [21], we find that for every $\epsilon > 0$ there exists an $M_\epsilon > 0$ such that for all $v \in C_0^\infty(\Omega)$,

$$\Big| \int_\Omega (\eta\Delta\eta + \frac{\eta^2}{4d^2}) v^2 dx \Big| \le \epsilon \int_\Omega \eta^2 |\nabla v|^2 dx + M_\epsilon \int_\Omega \eta^2 v^2 dx. \tag{5.2}$$

In the sequel we will establish the existence of a function $\psi_1 \in H_d^1(\Omega)$ which realizes the infimum in (5.1). To this end let w_k be a minimizing sequence normalized by $\int_\Omega \eta^2 w_k^2 dx = 1$. Then, using (5.2) we can easily obtain that the sequence w_k is bounded in H_d^1. Therefore there exists a subsequence still denoted by w_k such that it converges H_d^1 – weakly to ψ_1, and in addition we have the following strong convergence, for $k \to \infty$,

$$\int_\Omega (\eta\Delta\eta + \frac{\eta^2}{4d^2}) w_k^2 dx \to \int_\Omega (\eta\Delta\eta + \frac{\eta^2}{4d^2}) \psi_1^2 dx,$$

and

$$\int_\Omega \eta^2 w_k^2 dx \to \int_\Omega \eta^2 \psi_1^2 dx.$$

Using the lower semicontinuity of the gradient term in the numerator of (5.1) the result follows.
Step II: An auxiliary estimate. For $\delta > 0$ small enough we set

$$\mu_1(\Omega_\delta) := \inf_{v \in C_0^\infty(\Omega_\delta)} \frac{\int_{\Omega_\delta} \left(d|\nabla v|^2 - \Delta d \ \frac{v^2}{2} \right) dx}{\int_{\Omega_\delta} dv^2 dx} \ , \tag{5.3}$$

where $\Omega_\delta = \{x \in \Omega \text{ s.t. dist } (x, \partial\Omega) < \delta\}$.

We will show that

$$\mu_1(\Omega_\delta) \to +\infty, \qquad \text{as} \qquad \delta \to 0. \tag{5.4}$$

Our starting point is the inequality

$$\int_{\Omega_\delta} \left(|\nabla u|^2 - \frac{1}{4} \frac{u^2}{d^2} \right) dx \ge \frac{1}{8} \int_{\Omega_\delta} \frac{X^2(d)}{d^2} u^2 dx \ , \qquad u \in C_0^\infty(\Omega_\delta), \tag{5.5}$$

for any $0 < \delta \le \delta_0$, for some δ_0 small enough, where $X(t) := (1 - \ln t)^{-1}$. To prove this one starts with the obvious relation

$$0 \le \int_{\Omega_\delta} \Big| \nabla u - \left(\frac{\nabla d}{2d} - \frac{X \nabla d}{2d} \right) u \Big|^2 dx \ .$$

Expanding the square, integrating by parts and using the fact that $|d\Delta d|$ can be made arbitrarily small in Ω_δ, for δ sufficiently small, the result follows. Changing variable as usual by $u = d^{\frac{1}{2}} v$, inequality (5.5) is equivalent to

$$\int_{\Omega_\delta} \left(d|\nabla v|^2 - \Delta d \frac{v^2}{2} \right) dx \ge \frac{1}{8} \int_{\Omega_\delta} \frac{X^2(d)}{d} v^2 dx \ , \qquad v \in C_0^\infty(\Omega_\delta) \ . \tag{5.6}$$

Therefore

$$\frac{1}{8} \frac{X^2(\delta)}{\delta^2} \le \frac{X^2(\delta)}{\delta^2} \frac{\int_{\Omega_\delta} \left(d|\nabla v|^2 - \Delta d \frac{v^2}{2} \right) dx}{\int_{\Omega_\delta} \frac{X^2(d)}{d} v^2 dx} \le \frac{\int_{\Omega_\delta} \left(d|\nabla v|^2 - \Delta d \frac{v^2}{2} \right) dx}{\int_{\Omega_\delta} v^2 d \ dx},$$

from which (5.4) follows.
Step III: Asymptotics of ϕ_1. The lower bound $C_1 d^{1/2}(x) \le \phi_1(x)$ is a consequence of the maximum principle and is derived in Lemma 7 in [13].

We will obtain the upper bound using maximum principle in a suitably small neighborhood of the boundary. Let $\psi_1(x) = \phi_1(x)/d^{1/2}$. Then, for $Ev := -\mathrm{div}(d\nabla v) - \frac{\Delta d}{2} v - \lambda_1 dv$, we have that $E\psi_1 = 0$. Moreover, we have that

$$E(1 - Cd) = C - \frac{\Delta d}{2} + d\left(-\lambda_1 + \frac{3C}{2}\Delta d + C\lambda_1 d\right) \ge 0,$$

in Ω_δ for δ small enough and $C > 0$ big enough. We next choose $\beta > 0$ big enough so that

$$\psi_1(x) \le \beta(1 - Cd) \qquad \text{on} \qquad \partial\Omega_\delta.$$

Let $g(x) := \psi_1(x) - \beta(1 - Cd)$ and $g^+ := \max\{0, g\}$. We clearly have that

$$\int_{\Omega_\delta} g^+ E g \le 0,$$

from which it follows that

$$\frac{\int_{\Omega_\delta}(d|\nabla g^+|^2 - \frac{\Delta d}{2}(g^+)^2)dx}{\int_{\Omega_\delta} d(g^+)^2 dx} \le \lambda_1.$$

This contradicts (5.4) unless $g^+ = 0$ from which it follows that $\phi(x) \le \beta d^{1/2}(x)$.

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