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On Similarity Solutions of a Heat Equation with a Nonhomogeneous Nonlinearity

Stathis Filippas

Department of Mathematics, University of Crete, 71409 Heraklion, Greece

and

Achilles Tertikas

Department of Mathematics, University of Crete, 71409 Heraklion, Greece; Institute of Applied and Computational Mathematics, FORTH, 71110 Heraklion, Greece

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We are interested in positive radially symmetric solutions of the semilinear equation

$$\Delta w - \frac{y \cdot \nabla w}{2} - Aw + |y|^l w^p = 0, \qquad \text{in } \mathbb{R}^n, \quad n \geqslant 3,$$

where $p > 1$, $l > -2$ and $A \equiv \frac{l+2}{2(p-1)}$. This equation is satisfied by self-similar solutions of a semilinear heat equation. We prove existence and non existence of solutions for various values of the parameters l and p. When solutions exist we study their asymptotic behavior and discuss their uniqueness. Our proofs are based on various continuity, comparison and Pohozaev type arguments. © 2000 Academic Press

1. INTRODUCTION

We are interested in positive radially symmetric solutions of the semilinear equation:

$$\Delta w - \frac{y \cdot \nabla w}{2} - Aw + |y|^l w^p = 0, \qquad \text{in } \mathbb{R}^n, \quad n \geqslant 3, \tag{1.1}$$

where $p > 1$, $l > -2$ and $A \equiv \frac{l+2}{2(p-1)}$. Equation (1.1) is derived from the semilinear heat equation:

$$u_t = \Delta u + |x|^l u^p. \tag{1.2}$$

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It is shown in [2] that solutions of (1.2) that are initially bounded may become infinite in finite time; see also [11, 13] for various extensions. This equation can be thought of, as a perturbation, by a nonhomogeneous term, of the classical semilinear heat equation corresponding to the case $l=0$. In this last case the asymptotic behavior of blowing up solutions, at least when $1<p<\frac{n+2}{n-2}$, is described by special "backward self-similar solutions" of (1.2) of the form:

$$u(x,t)=(-t)^{-1/(p-1)}\,w(x/\sqrt{-t}), \qquad t<0. \tag{1.3}$$

Substitution of this into (1.2) yields (1.1) with $l=0$. This equation is by now well understood (cf [6, 3, 15, 5]).

Let us note that if we replace $-t$ with t in (1.3) we then obtain a new equation for w (differing from (1.1) in the sign of the second and third term) which describes the so called "forward self-similar solutions", related to the large time behavior of global solutions of (1.2). This equation has also received considerable attention (cf., e.g., [8, 4, 18]).

In the general case $l>-2$, standard scaling arguments suggest looking at special solutions of the form:

$$u(x,t)=(-t)^{-(l+2)/2(p-1)}\,w(x/\sqrt{-t}), \qquad t<0.$$

Plugging this into (1.2) we end up with the equation at hand (1.1). In analogy with the case $l=0$, we expect that the study of (1.1) is a reasonable first step towards the understanding of the behavior of blowing up solutions of (1.2).

Besides its connection with problem (1.2), Eq. (1.1) can be thought of, as a lower order perturbation of the well known elliptic problem (cf. [7]):

$$\Delta u+|x|^l\,u^p=0. \tag{1.4}$$

There is an extensive literature on equations related to (1.4), where questions about the existence, uniqueness or properties of the solutions are asked, for the Cauchy problem or for boundary value problems; see, e.g., [1, 9, 10, 17].

Associated to problem (1.2), as well as to its elliptic counterpart (1.4), are two critical exponents $1<p_c<p^c$ defined by:

$$p_c=\frac{n+2l+2}{n-2},$$

$$p^c=\begin{cases}\dfrac{(n-2)^2-2(l+2)(n+l)+2(l+2)\sqrt{(n+l)^2-(n-2)^2}}{(n-2)(n-10-4l)} & n>10+4l\\[2mm] +\infty & n\leqslant 10+4l.\end{cases}$$

When $1<p<p_c$ Eq. (1.4) has no nontrivial positive bounded solutions whereas for $p \geqslant p_c$ it has infinitely many radial regular solutions, see e.g., [7]. Moreover for $p > \frac{n+l}{n-2}$ it admits the following singular solution:

$$U(r) = Kr^{-(l+2)/(p-1)}, \qquad K \equiv \left(\frac{(l+2)(n-2)}{(p-1)^2} \left(p - \frac{n+l}{n-2} \right) \right)^{1/(p-1)}.$$

It is straightforward to check that U is also a singular solution of (1.1).

By standard regularity theory, bounded solutions of (1.1) belong to $C^2_{loc}(\mathbb{R}^n \backslash \{0\}) \cap C^{\alpha}_{loc}(\mathbb{R}^n)$ for any $\alpha \in (0, l+2)$. Thus, if $l \geqslant 0$ bounded solutions are always classical, whereas for $-2<l<0$ they are meant to be weak solutions which are continuous but in general non differentiable.

When looking for radially symmetric solutions, (1.1) can be written in divergence form as:

$$\frac{1}{\sigma}(\sigma w')' - Aw + r^l w^p = 0, \qquad \sigma(r) = r^{n-1} e^{-r^2/4}. \tag{1.5}$$

When $l \geqslant 0$ the natural conditions at the origin that complement Eq. (1.5) are $w(0) = a$ and $w'(0) = 0$. If $l<0$ however, due to the degeneracy of the equation (the nonlinear term of the equation becomes infinite at the origin) the last condition is being replaced by

$$(\sigma w')(0) := \lim_{r \downarrow 0} \sigma(r)\, w'(r) = 0,$$

which as we shall see in Section 2 implies that $\lim_{r \downarrow 0} rw'(r) = 0$.

As in the case $l=0$, the properties of solutions of (1.1) depend crucially on the range of the exponent p. In fact we will prove:

THEOREM A. (a) *Let $1<p<p_c$. Then, if $-2<l<0$ Eq.* (1.1) *admits a bounded positive radially symmetric and decreasing solution, whereas if $l>0$ it has no bounded solutions.*

(b) *Let $p_c<p<p^c$. Then* (1.1) *has infinitely many positive bounded radial solutions.*

If $l=0$ and $1<p<p_c$ it is known that (1.1) admits the constant $(p-1)^{-1/(p-1)}$ as its unique positive solution (cf. [6]). In contrast, if $l \neq 0$ Eq. (1.1) admits a nonconstant positive bounded solution or no bounded solutions at all, depending on l. When $p_c<p<p^c$ however, the situation is quite similar as in the case $l=0$ (cf. [3, 15]). In addition to this existence result, we obtain various information on the asymptotic properties of solutions of (1.1).

An interesting question that we leave open in this work is whether the solution constructed for $1<p<p_c$ and $-2<l<0$ is unique or not. We conjecture that it is unique.

In a large class of problems, the uniqueness of positive solutions with "finite energy" (that is $\int |\nabla u|^2<\infty$), in $\mathbb{R}^n$ is strongly linked to the uniqueness of positive solutions of the corresponding Dirichlet problems (cf [1, 9, 10, 17, 14]). In this direction, let B_R be a ball of radius R centered at the origin, and consider the Dirichlet problem for (1.1) in B_R with zero boundary conditions. We then show:

THEOREM B. *Let $1<p<p_c$ and $-2<l<0$. There exists an R_0 depending on p, n, l, such that the Dirichlet problem for* (1.1) *in B_R has a unique bounded radial solution if $0<R<R_0$.*

The method we use (based on ideas of [10]) breaks down for large R. This is not accidental since we do not have uniqueness for large R, at least under some assumptions on l.

In this work we focus our attention to radial solutions only. The existence or not, of nonradial solutions of (1.1) remains an interesting question.

Most of the ideas we use, work for the special case $l=0$ and in many occasions simplify the existing arguments. The rest of the paper is organized as follows. In the first three sections we consider the case $-2<l<0$ and $1<p<p_c$. At first we show the existence of a radially decreasing solution. We then study the asymptotic behavior of this solution, and finally in Section 3 we discuss its uniqueness and prove Theorem B. In Section 4 we show the nonexistence part of Theorem A(a) whereas in the last section we consider the super critical case and prove Theorem A(b).

2. EXISTENCE OF A RADIALLY DECREASING SOLUTION

In this section we consider the case $-2<l<0$ and $1<p<p_c$. We look for positive radially symmetric solutions of the semilinear equation:

$$w''+\left(\frac{n-1}{r}-\frac{r}{2}\right)w'-Aw+r^l w^p=0, \tag{2.1}$$

or equivalently,

$$(\sigma w')'-A\sigma w+\sigma r^l w^p=0, \qquad \sigma=r^{n-1}e^{-r^2/4},$$

with initial condition $w(0)=a>0$.

We will use a shooting argument. Let I_+ denote the set of the initial values $a = w(0) > 0$, for which the corresponding solutions take on a positive minimum before hitting the r-axis (if they ever hit). We also denote by I_- the set of initial values for which the solutions are monotone decreasing until they cross the r-axis. By continuity, both sets are easily seen to be open. If we prove that they are nonempty, we then obtain the existence of at least an initial value in $\mathbf{R}^+ - I_+ \cup I_- \neq \emptyset$ to which there corresponds a radially symmetric and decreasing solution of (2.1).

To prove that I_+ is nonempty we will look at low shootings that is, small values of a. To this end we set:

$$u(r, a) = w(r)/a$$

so that

$$(\sigma u')' - A\sigma u + \sigma r^l a^{p-1} u^p = 0, \qquad u(0, a) = 1. \tag{2.2}$$

Using a continuous dependence argument, we will show that for small a solutions of (2.2) stay close to the solution of the limiting linear problem:

$$(\sigma u_L')' - A\sigma u_L = 0, \qquad u_L(0) = 1,$$

on compact intervals $[0, R]$.

Lemma 2.1. *Let $l > -2$. For every $\varepsilon \in (0, 1)$ and $R > 0$, there exists an $a_0 > 0$ such that for $0 < a < a_0$ we have that:*

$$|u(r, a) - u_L(r)| \leqslant \varepsilon, \qquad \text{in } [0, R].$$

Proof. After integrating twice, using $(\sigma u')(0) = 0$, one can easily check that Eq. (2.2) may be recast in the following integral form:

$$u(r, a) = 1 + \int_0^r K(r, s)\, u(s, a)(A - s^l a^{p-1} u^{p-1}(s, a))\, ds, \tag{2.3}$$

with

$$K(r, s) \equiv \sigma(s) \int_s^r \sigma^{-1}(\lambda)\, d\lambda, \qquad 0 \leqslant s \leqslant r.$$

Notice that

$$0 \leqslant K(r, s) = s^{n-1} e^{-s^2/4} \int_s^r \lambda^{-(n-1)} e^{\lambda^2/4} \, d\lambda$$

$$\leqslant s^{n-1} e^{r^2/4} \int_s^r \lambda^{-(n-1)} \, d\lambda = \frac{e^{r^2/4}}{n-2} s^{n-1} \left(\frac{1}{s^{n-2}} - \frac{1}{r^{n-2}} \right)$$

$$\leqslant \frac{e^{r^2/4}}{n-2} s \equiv C(r) s.$$

Therefore, the integrand in the right hand side of (2.3) is of the order $O(s^{1+l})$ for s near zero and the integral is well defined for $l > -2$.

The integral form of the corresponding linear problem is:

$$u_L(r) = 1 + A \int_0^r K(r, s) \, u_L(r) \, ds. \tag{2.4}$$

Subtract (2.4) from (2.3) to get:

$$u(r, a) - u_L(r) = A \int_0^r K(r, s)(u(s, a) - u_L(s)) \, ds$$

$$- a^{p-1} A \int_0^r K(r, s) \, s^l u^{p-1}(s, a) \, ds. \tag{2.5}$$

Fix an $\varepsilon \in (0, 1)$ and an $R > 0$. Let us take an $a_0 > 0$ small enough (we will precise it later) so that

$$|u(r, a) - u_L(r)| \leqslant \varepsilon, \qquad \text{in } [0, r_0], \tag{2.6}$$

for some $r_0(a_0)$. This is of course always possible since $u(0, a) = u_L(0) = 1$ and they are both continuous. We will show that by choosing a_0 sufficiently small we can ensure that $r_0 = R$.

From (2.5), (2.6) and the estimate of $K(r, s)$ we have for $r \in (0, r_0)$:

$$|u(r, a) - u_L(r)| \leqslant AC(R) \int_0^r s \, |u(s, a) - u_L(s)| \, ds$$

$$+ a^{p-1} C(R) \int_0^r s^{1+l} (1 + \sup_{(0, r_0)} |u_L(r)|)^p \, ds$$

$$\geqslant AC(R) \int_0^r s \, |u(s, a) - u_L(s)| \, ds$$

$$+ a_0^{p-1} \frac{C(R) \, R^{2+l}}{l+2} (1 + \sup_{(0, R)} |u_L(r)|)^p.$$

Applying now Gronwall's inequality we end up with:

$$|u(r,a)-u_L(r)| \leqslant a_0^{p-1}\,\frac{C(R)\,R^{2+l}}{l+2}\,(1+\sup_{(0,R)}|u_L(r)|)^p\, e^{(1/2)\,AC(R)\,R^2},$$

$$\text{in } (0, r_0). \tag{2.7}$$

It follows that by choosing a_0 small enough so that the right hand side of (2.7) is less than ε, we can take $r_0 = R$ and the lemma is proved. ▮

Remark 2.1. It follows from (2.3) that $w'(r) = O(r^{1+l})$ as $r \to 0$, and therefore $rw'(r) = O(r^{2+l}) \to 0$ as $r \to 0$, as mentioned in the Introduction. In addition we have that $w''(r) = O(r^l)$.

As a consequence of Lemma 2.1 we have:

LEMMA 2.2. *I_+ contains a (positive) neighborhood of zero.*

Proof. It is easy to see that $u(r, a)$ is a decreasing function of r, near zero, for a small but fixed. Indeed, the integrand in (2.3) is negative for r near zero. Consequently $u(r', a) < 1$ for some r'.

On the other hand, for a, R, as in Lemma 2.1, $u(R, a)$ will stay close to $u_L(R)$ and since $u_L(r)$ is an increasing function it follows that $u(R, a) > 1$.

We conclude that $u(r, a)$ takes on a minimum in $[0, R]$ while staying positive. Since the same holds true for $w(r, a)$ the Lemma is proved. ▮

We next see what happens for high shootings, that is, for large values of $w(0)$. We use the following normalization:

$$u(r,\beta) \equiv w(\beta^{-1/2A} r)/\beta,$$

so that $u(r, \beta)$ satisfies:

$$u''(r,\beta) + \frac{n-1}{r}\,u'(r,\beta) + r^l\,|u(r,\beta)|^{p-1}\,u(r,\beta)$$

$$-\beta^{-1/A}\left(\frac{r}{2}\,u'(r,\beta) + Au(r,\beta)\right) = 0, \tag{2.8}$$

with $u(0, \beta) = 1$. We will compare solutions of (2.8) with the solution of the limiting problem ($\beta = \infty$):

$$u_*''(r) + \frac{n-1}{r}\,u_*'(r) + r^l\,|u_*(r)|^{p-1}\,u_*(r) = 0, \qquad u_*(0) = 1. \tag{2.9}$$

It is known that problem (2.9), with $l > -2$ and $1 < p < p_c$, has a unique solution in $(0, R_1)$ for some $R_1 > 0$ which is decreasing in this interval, and

$u_*(R_1)=0$ (cf. [12]). This solution can of course be extended to $(0, R_1+\delta)$ for some $\delta>0$ so that $u_*(r)<0$ and $u'_*(r)<0$ in $(R_1, R_1+\delta)$.

LEMMA 2.3. *Let $l>-2$. For every $\varepsilon>0$ there exists a $\beta_0>0$ such that for $\beta>\beta_0$:*

$$|u(r,\beta)-u_*(r)|<\varepsilon \qquad \text{in } [0, R_1+\delta].$$

Proof. We will use an argument quite similar to that used in Lemma 2.1. The integral form of the equation:

$$u''+\frac{n-1}{r}u'=f, \qquad u(0)=1,$$

is easily seen to be:

$$u(r)=1+\int_0^r L(r,s)\, f(s)\, ds,$$

with:

$$L(r,s)\equiv\frac{s}{n-2}\left(1-\frac{s^{n-1}}{r^{n-1}}\right), \qquad 0\leqslant s\leqslant r.$$

We denote by C a constant not necessarily the same in each appearance, which may depend on n, p, $R_1+\delta$, and $\sup_{(0, R_1+\delta)} u_*(r)$ but not on β.

The following estimates are easily deduced from the definition of the kernel $L(r,s)$:

$$0\leqslant L(r,s)\leqslant Cs, \qquad \left|\frac{\partial L(r,s)}{\partial s}\right|<C. \tag{2.10}$$

For some $\varepsilon\in(0,1)$ and $r_0>0$ we have the *a priori* estimate:

$$|u(r,\beta)-u_*(r)|<\varepsilon \qquad \text{in } [0, r_0). \tag{2.11}$$

Using the integral form of both $u(r,\beta)$ and $u_*(r)$ we calculate:

$$u(r,\beta)-u_*(r)=\int_0^r L(r,s)\, s^l(|u_*(s)|^{p-1}\, u_*(s)-|u(s,\beta)|^{p-1}\, u(s,\beta))\, ds$$
$$+\frac{1}{2}\beta^{-1/A}\int_0^r L(r,s)\, su'(s,\beta)\, ds+A\beta^{-1/A}\int_0^r L(r,s)\, u(s,\beta)\, ds. \tag{2.12}$$

Using (2.10), (2.11) and standard inequalities we estimate the terms of the right hand side of (2.12) in the interval $(0, r_0)$.

$$\left| \int_0^r K(r, s)\, s^l (|u_*(s)|^{p-1} u_*(s) - |u(s, \beta)|^{p-1} u(s, \beta))\, ds \right|$$
$$\leqslant C \int_0^r s^{1+l} |u_*(s) - u(s, \beta)|\, ds.$$

For the third term we have that:

$$\left| \int_0^r L(r, s)\, u(s, \beta)\, ds \right| \leqslant C.$$

Finally for the second term we get

$$\int_0^r L(r, s)\, su'(s, \beta)\, ds = -\int_0^r \frac{\partial L(r, s)}{\partial s} su(s, \beta)\, ds - \int_0^r L(r, s)\, u(s, \beta)\, ds,$$

therefore

$$\left| \int_0^r L(r, s)\, su'(s, \beta)\, ds \right| \leqslant C.$$

Putting everything together we have:

$$|u(r, \beta) - u_*(r)| \leqslant C \int_0^r s^{1+l} |u(s, \beta) - u_*(s)|\, ds + \beta^{-1/A} C.$$

Applying now Gronwall's inequality we end up with:

$$|u(r, \beta) - u_*(r)| \leqslant C\beta^{-1/A}, \qquad \text{in } (0, r_0),$$

for some constant C independent of β. We can now complete the proof as in Lemma 2.1. We omit further details. ∎

Remark 2.2. We note that Lemmas 2.1 and 2.3 hold true for any $p > 1$ and $l > -2$.

As a consequence of the above Lemma we have:

LEMMA 2.4. *I_- contains a neighborhood of infinity.*

Proof. For β sufficiently large, it follows from Lemma 1.3 that $u(r, \beta)$ will cross the r-axis at a point R' near R_1. It remains to show that $u(r, \beta)$ is also decreasing as long as it is positive.

Suppose $u(r, \beta)$ takes on a positive minimum at some point $r_m \in (0, R')$ At this point we have:

$$u(r_m, \beta)(r_m^l u^{p-1}(r_m, \beta) - \beta^{-1/A} A) = u''(r_m, \beta) \leqslant 0$$
$$\Rightarrow r_m^l u^{p-1}(r_m, \beta) \leqslant \beta^{-1/A} A.$$

Since β is large, r_m stays out of a neighborhood of zero, say $(0, \rho)$. In the interval $(\rho, R_1 + \delta)$ Eq. (2.8) contains no singular term, and we can use standard arguments to show that $u'(r, \beta)$ stays arbitrarily close to $u'_*(r)$ in $(\rho, R_1 + \delta)$ for large β (e.g., differentiate the equation and use an argument as in Lemma 2.1. Since $u'_*(r) < 0$ in $(\rho, R_1 + \delta)$ the same is true for $u'(r, \beta)$. Consequently, $u(r, \beta)$ (and $w(r, \beta)$) is decreasing as required, and the lemma is proved. ∎

Remark 2.3. We have thus shown that for large β, $w(r, \beta)$ is monotone and crosses the r-axis at a point $R_\beta \approx \beta^{-1/2A} R_1 \to 0$ as $\beta \to +\infty$. On the other hand, it follows from the previous analysis that if $\hat{\beta} \equiv \inf I_-$ then $w(r, \hat{\beta})$ is a decreasing solution of (2.1). We conclude by continuity that, for any $R \in (0, +\infty)$ the Dirichlet problem for (2.1) in a ball B_R centered at the origin and of radius R, admits a radially symmetric and decreasing solution.

3. ASYMPTOTIC BEHAVIOR OF POSITIVE SOLUTIONS

In the previous section we showed the existence of a radially decreasing solution in the case where $-2 < l < 0$ and $1 < p < p_c$. Here we will study the asymptotic properties of positive solutions when p and l are as before. At first we will derive the asymptotic behavior of bounded solutions and then we will discuss the case of unbounded solutions. Our first goal is to prove the following:

PROPOSITION 3.1. *Let $w(r)$ be a bounded, positive and radially symmetric solution of* (2.1). *Then, for large values of r, $w(r)$ is a decreasing function and*

$$\lim_{r \to +\infty} w(r)\, r^{2A} = c_0,$$

for some positive constant c_0. In addition, $w(r)\, r^{2A}$ is an increasing function of r in $(0, +\infty)$.

To prove this proposition we will use several lemmas. At first we have:

LEMMA 3.2. *Every bounded solution of* (2.1) *is eventually monotone decreasing in r.*

Proof. Equation (2.1) can be written as:

$$(\sigma w')' = \sigma w(A - r^l w^{p-1}). \tag{3.1}$$

Given that w is bounded, we see that for large r, say $r > R$, we have that $(\sigma w')'$ is positive, that is, $\sigma w'$ is an increasing function of r. If w' were positive at some point $r_0 > R$, then $\sigma w'$ would have stayed positive and increasing for all $r > r_0$. But then, since $\sigma(r)$ is positive and decreases to zero, we would have that w' goes to infinity, a contradiction. Consequently, $w'(r) < 0$ for $r > R$. ∎

We next show that the solution decreases to zero.

LEMMA 3.3. $\lim_{r \to +\infty} w(r) = 0$.

Proof. Since $w(r)$ is decreasing and positive it has a nonnegative limit. Let us call this limit c and assume that it is strictly positive $c > 0$. Integrating (3.1) along a sequence of points $\{r_n\} \to +\infty$ at which $|w'(r_n)| < 1$, we get:

$$\sigma w'(r) = \int_r^{+\infty} \sigma w(A - s^l w^{p-1})\, ds.$$

Using this expression and L'Hopital's rule we compute that

$$rw'(r) \to 2Ac > 0, \qquad \text{as} \quad r \to +\infty,$$

which is clearly a contradiction. Thus, $c = 0$. ∎

We want to find the rate at which $w(r)$ tends to zero. Following [5] we have:

LEMMA 3.4. *For large values of r we have that:*

$$w(r) \leqslant Cr^{-2A},$$

for some constant C, independent of r.

Proof. We first prove a weaker estimate. Since $w(r)$ decreases to zero, for any $\mu < A$ and r large we have:

$$w'' - \frac{r}{2} w' - \mu w \geqslant 0.$$

On the other hand for large values of r, say $r \geqslant r_0$, the function $W = Mr^{-2\theta}$, with $M > 0$ and $\theta < \mu$ satisfies:

$$W'' - \frac{r}{2} W' - \mu W \leqslant 0.$$

Taking M large enough so that $W(r_0) > w(r_0)$ we conclude by comparison that

$$w(r) \leqslant W(r) = Mr^{-2\theta}, \tag{3.2}$$

for any $\theta < A$ and $r \geqslant r_0$.

We next improve this estimate. Set $z = wr^{2A}$. Then, z satisfies the equation:

$$z'' - \frac{1}{2} rz' + \frac{n-1-4A}{r} z' + \frac{z(z^{p-1} - 2A(n-2-2A))}{r^2} = 0. \tag{3.3}$$

From (3.2) and the fact that $w'(r) < 0$ for large r, we easily get that for any $\delta > 0$:

$$z(r) < Cr^{\delta}, \qquad \text{and} \qquad z'(r) \leqslant Cr^{\delta - 1}.$$

It then follows from (3.3) that the quantity $(z''/r - z'/2)$ is integrable near infinity. Integrating by parts this quantity in (r, ∞) we find that $\lim_{r \to +\infty} z(r)$ exists and the lemma is proved. ∎

We are now ready to complete the proof of Proposition 3.1:

End of proof of Proposition 3.1. We make the following change of variables:

$$v(s) = w(r)\, r^{2A}, \qquad s = \ln r,$$

so that $v(s)$ satisfies in $(-\infty, +\infty)$:

$$v'' - (\theta + \tfrac{1}{2} e^{2s})\, v' + cv + v^p = 0, \tag{3.4}$$

with

$$\theta \equiv 4A + 2 - n > 0, \quad (1 < p < p_c) \qquad \text{and} \qquad c \equiv 2A(2A + 2 - n).$$

We define the energy functional:

$$E[v](s) = \frac{1}{2} v'^2 + \frac{c}{2} v^2 + \frac{1}{p+1} v^{p+1}.$$

It is straightforward to verify that:

$$v(-\infty) = 0, \qquad E[v](-\infty) = 0, \qquad \frac{dE}{ds} = \left(\theta + \frac{1}{2} e^{2s}\right) v'^2 > 0.$$

We conclude that $E[v](s) > 0$ for all s.

We now show that $v(s)$ is an increasing function of $s \in (-\infty, +\infty)$. It is enough to show that $v(s)$ does not have a maximum at any point $s \in (-\infty, +\infty)$. Suppose on the contrary it has a maximum at a point s_0. There are two possibilities: either $v(s)$ takes on a positive minimum after s_0 or else $v(s)$ decreases to a nonnegative constant at infinity. In both cases we get a contradiction. For instance, let us assume that $v(s)$ takes a minimum at a point $s_1 > s_0$. Using the monotonicity of the energy, we get:

$$0 < E[v](s_0) = v_0^2 \left(\frac{c}{2} + \frac{1}{p+1} v_0^{p-1} \right) \leqslant v_1^2 \left(\frac{c}{2} + \frac{1}{p+1} v_1^{p-1} \right) = E[v](s_1),$$

which is impossible since $v_1 < v_0$. Similarly we exclude case (ii). Consequently, $v(s)$ is increasing, and since it is also bounded (from the previous Lemma), it converges to a positive constant. ∎

Concerning the large time behavior of any positive shooting (including the ones that might be unbounded at infinity) we show the following result, which will be used in the next section.

PROPOSITION 3.5. *Every positive and radially symmetric solution of* (2.1) *satisfies for large* r:

$$w(r) \leqslant C_0 r^{-l/(p-1)}, \tag{3.5}$$

for some positive constant depending only on p.

Before proving this result we present an auxiliary lemma:

LEMMA 3.6. *There is no positive solution of* (2.1) *such that for* r *large, say* $r \geqslant r_0$:

$$w(r) > \left(\frac{1}{p-1} \right)^{1/(p-1)} r^{-l/(p-1)}. \tag{3.6}$$

Proof. Assuming that (3.6) holds we will reach a contradiction. Using the w-equation and (3.6) we get for $r \geqslant r_0$:

$$(\sigma w')' = \sigma w (A - r^l w^{p-1}) < \frac{l}{2} \sigma w^p r^l < 0. \tag{3.7}$$

Thus, $\sigma w'$ is a decreasing function. If we assume that $w'(r_1) < 0$ for some $r_1 > r_0$ then $w' < 0$ for all $r > r_1$ contradicting (3.7). We conclude that $w(r)$ is an increasing function for $r \geqslant r_0$.

Integrating (3.7) and using the monotonicity of w we get:

$$\sigma w'(r) > -\frac{l}{2} w^p(r) \int_r^{+\infty} \sigma(s)\, s^l\, ds,$$

or, using L'Hopital's rule,

$$\frac{w'}{w^p} > -\frac{l}{2} \frac{\int_r^{+\infty} \sigma(s)\, s^l\, ds}{\sigma(r)} \approx -lr^{l-1}.$$

It then follows that:

$$\left(-\frac{1}{p-1}\frac{1}{w^{p-1}} + r^l\right)' \geqslant 0.$$

Since the quantity inside the parenthesis tends to zero it should be negative, whence we obtain that $w^{p-1} r^l \leqslant \frac{1}{p-1}$, which contradicts (3.6). ∎

We are now ready prove Proposition 3.5.

Proof of Proposition 3.5. We will use the following change of variables:

$$s = \ln r, \qquad q(s) = w(r)\, r^{l/(p-1)}.$$

We then obtain the following equation for $q(s)$:

$$e^{-2s} q'' - \left(\frac{1}{2} - \theta_1 e^{-2s}\right) q' - \left(\frac{1}{p-1} - \theta_2 e^{-2s}\right) q + q^p = 0, \tag{3.8}$$

with

$$\theta_1 \equiv n - 2 - \frac{2l}{p-1} > 0, \qquad \theta_2 \equiv -\frac{l}{p-1}\left(n - 2 - \frac{l}{p-1}\right) > 0.$$

Let $\bar{q}$ be a constant such that:

$$\bar{q} > q_0 \equiv \left(\frac{p+1}{2(p-1)}\right)^{1/(p-1)} > \left(\frac{1}{p-1}\right)^{1/(p-1)} \equiv k.$$

We will show that $q(s)$ cannot oscillate about $\bar{q}$ for large values of s. Since it cannot stay above $\bar{q}$ (from Lemma 3.6) it follows that $q(s) \leqslant \bar{q}$ which is the same as (3.5).

We define the energy functional:

$$E[q](s) = \frac{1}{2} e^{-2s} q'^2 - \frac{1}{2}\left(\frac{1}{p-1} + \theta_2 e^{-2s}\right) q^2 + \frac{1}{p+1} q^{p+1}.$$

A straightforward calculation, using the q-equation, yields:

$$\frac{dE}{ds}=\left(\frac{1}{2}-(1+\theta_1+\theta_2)\,e^{-2s}\right)q'^2+e^{-2s}\theta_2(q-q')^2.$$

We conclude that for large values of s the energy E is monotone increasing.

Let us assume that $q(s)$ oscillates about $\bar{q}$ in order to get a contradiction. Then q will have an infinite sequence of maxima and minima which we denote by q_{M_i} and q_{m_j} respectively. Clearly, we have $q_{M_i}>\bar{q}$, whereas it follows easily from the q-equation that $q_{m_j}<k$. An easy computation then shows that for s large enough, $E[q_{M_i}]>0$ and $E[q_{m_i}]<0$ which clearly contradicts the monotonicity of E. ▮

Remark 3.1. Proposition 3.5 is valid for any $p>1$ and any positive radially symmetric solution, even the singular ones.

Remark 3.2. Let us consider the special case $l=0$. By slightly adapting the arguments in Lemma 3.6, it is not difficult to check that Proposition 3.5 holds true in this case as well. Thus, all radial positive solutions of Eq. (1.1) are eventually bounded. Since we know that (1.1) admits a unique bounded positive solution $w=(p-1)^{-1/(p-1)}$ (cf. [6]) we conclude that all shootings $w(0)=a<+\infty$ cross the r-axis, (except when $a=(p-1)^{-1/(p-1)}$), if $l=0$.

4. ABOUT UNIQUENESS

A natural question is the question of uniqueness of the solution constructed. Numerical experiments suggest that Problem (1.1) admits a unique radial solution, something that we have not been able to prove at this stage. Let us however consider the Dirichlet problem:

$$\Delta w-\frac{y\cdot\nabla w}{2}-Aw+|y|^l\,w^p=0,\qquad w>0\quad\text{in } B_R,\qquad w=0\quad\text{on }\partial B_R,\tag{4.1}$$

where B_R denotes the ball in $\mathbb{R}^n$ of radius R centered at the origin. We know that for any $R>0$ Eq. (4.1) admits a radially symmetric and decreasing solution (cf. Remark 2.3). On the other hand we expect that in general (4.1) will have non monotone radial solutions as well. Indeed, let us consider a shooting $w(r,a)$ with $a\in I_+$. Then, at least when l is very close to zero we can get by continuity that $w(r,a)$ will cross the r-axis at some point r_0 (cf. Remark 3.2). Thus, at B_{r_0} we get two different solutions.

Using ideas from [10] we show:

THEOREM 4.1. *Let* $-2<l<0$ *and* $1<p<p_c$. *There exists an* R_0 *depending on* p, n, l, *such that if* $0<R\leqslant R_0$ *Eq.* (4.2) *has a unique bounded radial solution.*

As we have seen we can write Eq. (4.1) as:

$$(\sigma w')' - A\sigma w + \sigma r^l w^p = 0, \qquad (\sigma w')(0) = w(R) = 0. \tag{4.2}$$

To prove this theorem we will use several lemmas. At first we have:

LEMMA 4.2. *If* w, $\bar{w}$ *are two positive solutions of* (4.2) *they intersect each other at least once.*

Proof. Multiply the w-equation by $\bar{w}$ and integrate by parts. Then, do the same with the $\bar{w}$-equation and subtract the two expressions to get:

$$\int_0^R w\bar{w}(w^{p-1} - \bar{w}^{p-1})\,\sigma r^l = 0.$$

If we assume that, say, $w(r) > \bar{w}(r)$ we get an obvious contradiction. ∎

We next show:

LEMMA 4.3. *Let* $w(0) > \bar{w}(0)$. *Then* $(\frac{w}{\bar{w}})' < 0$ *until at least* w *and* $\bar{w}$ *intersect each other.*

Proof. Let $W(r) = w'\bar{w} - w\bar{w}'$. It is straightforward to verify that $(\sigma W)(0) = 0$ and (using the equation):

$$(\sigma W)' = -\sigma r^l w\bar{w}(w^{p-1} - \bar{w}^{p-1}).$$

It then follows that $W(r) < 0$ as long as $w > \bar{w}$ and the lemma is proved. ∎

These are standard facts that hold true for any $R > 0$, including the case $R = \infty$. To proceed further we make the following change of variables:

$$u = wf \qquad \text{with} \quad f(r) = r^{(l+2(n-1))/(p+3)} e^{-r^2/2(p+3)},$$

so that u satisfies the equation:

$$B(r)\,u'' + C(r)\,u' + G(r)\,u + u^p = 0, \tag{4.3}$$

with

$$B(r)=r^{-l}f^{p-1}, \qquad C(r)=r^{-l}f^{p-1}\left(\frac{\sigma'}{\sigma}-2\frac{f'}{f}\right),$$

$$G(r)=r^{-l}f^{p-1}\left(2\left(\frac{f'}{f}\right)^2-\frac{f''}{f}-\frac{\sigma'}{\sigma}\frac{f'}{f}-A\right).$$

The choice of f made above, is such that $B'(r)=2C(r)$ so that if we define the energy functional:

$$E[u](r)=\frac{1}{2}Bu'^2+\frac{1}{2}Cu^2+\frac{1}{p+1}u^{p+1},$$

the following simple identity holds:

$$\frac{dE}{dr}=\frac{1}{2}G'(r)\,u^2. \tag{4.4}$$

It is straightforward to check that $E[u](0)=0$. Integrating then (4.4), we get:

$$2E[u](r)=Bu'^2+Cu^2+\frac{2}{p+1}u^{p+1}=\int_0^r G'(s)\,u^2. \tag{4.5}$$

The properties of the function G will play a crucial role in the method we use. Since G is known explicitly, we just calculate to get:

$$G(r)=-\frac{r^{-2(p_c-p)(n-2)/(p+3)}}{(p+3)^2}\,e^{-r^2(p-1)/2(p+3)}(ar^4-br^2+c),$$

where:

$$a=\frac{p+1}{2}>0, \qquad b=\frac{1}{2}(l+2n-2)(p-1)+(p+3)(n-A(p+3)),$$

$$c=(l+2n-2)((n-2)(p+3)-(l+2n-2))>0.$$

We then have that $G(r)\to-\infty$ as $r\to 0$ and $0>G(r)\to 0$ as $r\to\infty$. If we differentiate G we will get an expression of the form $G'=Z(r)\,P_3(r^2)$, where $Z(r)$ is a function of constant sign and $P_3(r^2)$ is a cubic polynomial in r^2. We conclude that either G is monotone increasing in $(0,+\infty)$ or else, it has exactly one local maximum $r_0>0$ and one local minimum $R_0>r_0>0$, so that G increases in $(0, r_0)$ and (R_0, ∞) and decreases in (r_0, R_0). For reasons that we will explain later (cf Remark after the proof of the theorem), $G(r)$ is not monotone for the current range of p and l, therefore the points r_0 and R_0 always exist.

In fact, the point R_0 defines the maximum interval $(0, R_0)$ for which we will prove uniqueness for the Dirichlet problem (4.2).

From now on we consider Eq. (4.2) in the interval $(0, R_0)$. It will be clear that the same arguments work for any $R \leqslant R_0$.

At first we notice that because of (4.4) and the discussion above, the energy $E[u](r)$ is increasing in $(0, r_0)$ and then decreases in (r_0, R_0) until it reaches the value $E[u](R_0) = Bu'^2 > 0$. We therefore conclude that E is positive in $(0, R_0)$.

We next show:

LEMMA 4.4. *If* $w, \bar{w}$ *are two positive solutions of* (4.2) *they intersect each other exactly once in* $(0, R_0)$. *As a consequence,* $w/\bar{w} = u/\bar{u}$ *is monotone in* $(0, R_0)$.

Proof. From Lemma 4.2 they intersect at least once, so it is enough to show that they do not intersect for a second time. Clearly, it is equivalent to show that this property holds true for $u = wf$ and $\bar{u} = \bar{w}f$.

Let r_0 be, as before, the point at which G attains its maximum in $(0, R_0)$. We set:

$$\mu = \frac{u(r_0)}{\bar{u}(r_0)}.$$

We assume for definiteness that $w(0) > \bar{w}(0)$ which by Lemma 4.3 implies that initially $(\frac{u}{\bar{u}})' < 0$.

Let us suppose that u and $\bar{u}$ intersect each other for a second time in order to get a contradiction. By the mean value theorem there will be a (first) point τ between the two intersection points of u and $\bar{u}$ at which:

$$\left(\frac{u(\tau)}{\bar{u}(\tau)}\right)' = 0, \qquad \text{or,} \quad u'(\tau) = \frac{u(\tau)}{\bar{u}(\tau)}\,\bar{u}'(\tau). \tag{4.6}$$

We may assume that $\tau > r_0$ (the other case is similar and simpler). We clearly have:

$$u(r) > \mu\bar{u}(r), \qquad \text{for} \quad r \in (0, r_0), \quad \text{and} \quad u(r) < \mu\bar{u}(r), \qquad \text{for} \quad r \in (r_0, R_0). \tag{4.7}$$

We now use the energy identity (4.5) with $r=\tau$. Multiply the $\bar{u}$-identity by μ^2 and subtract it from the u-identity to get:

$$B(u'^2-\mu^2\bar{u}'^2)+G(u^2-\mu^2\bar{u}^2)+\frac{2}{p+1}(u^{p+1}-\mu^2\bar{u}^{p+1})$$

$$=\int_0^\tau G'(s)(u^2(s)-\mu^2\bar{u}^2(s))\,ds>0. \tag{4.8}$$

To see why the right hand side is positive, we just split the integral from 0 to r_0 and from r_0 to R_0 and use the definition of r_0 and (4.7). Concerning the left hand side we can use (4.6) to rewrite it as:

$$LHS=\frac{(u^2-\mu^2\bar{u}^2)}{\bar{u}^2}\left(B\bar{u}'^2+G\bar{u}^2+\frac{2\bar{u}^2}{p+1}\frac{u^{p+1}-\mu^2\bar{u}^{p+1}}{u^2-\mu^2\bar{u}^2}\right).$$

Using (4.7) and the fact that $u(\tau)<\bar{u}(\tau)$ the last term in the big parenthesis above is easily seen to be greater than $\frac{2}{p+1}\bar{u}^{p+1}$. On the other hand, the factor outside the parenthesis is negative because of (4.7). Consequently we have:

$$LHS<2\,\frac{(u^2-\mu^2\bar{u}^2)}{\bar{u}^2}\,E[u](\tau)<0,$$

which contradicts (4.8). We conclude that u and $\bar{u}$ cannot intersect for a second time, as desired.

As a matter of fact we have shown that the ratio $(u/\bar{u})'=(w/\bar{w})'$ is different from zero in $(0, R_0)$. Hence, the last statement of the lemma follows. ∎

We can now give the proof of the theorem.

Proof of Theorem 4.1. We will use the same argument by contradiction as in the previous lemma. Let r_0 and μ be as before. We will use the identity (4.5) with $r=R_0$. Multiply the $\bar{u}$-identity with μ^2 and subtract it from the u-identity to get:

$$u'^2(R_0)-\mu^2\bar{u}'^2(R_0)=\int_0^{R_0} G'(s)(u^2(s)-\mu^2\bar{u}^2(s))\,ds>0 \tag{4.9}$$

The right hand side is positive by the same reasoning as in Lemma 4.4. Assuming, as before, for definiteness that $w(0)>\bar{w}(0)$ it follows from

Lemma 4.4 that we also have $(\frac{u}{\bar{u}})' < 0$ in $(0, R_0)$. Using L'Hopital's rule we get:

$$\frac{u'(R_0)}{\bar{u}'(R_0)} = \frac{u(R_0)}{\bar{u}(R_0)} \leqslant \mu \Rightarrow u'^2(R_0) - \mu^2 \bar{u}'^2(R_0) \leqslant 0,$$

which contradicts (4.9). ∎

Remark 4.1. It is clear that if the function $G(r)$ were monotone decreasing in $(0, \infty)$ for some choice of p, n, l then by the previous argument, we would obtain uniqueness for all R including the case $R = +\infty$. To see that this is impossible let us consider a non monotone shooting for Eq. (4.2). If it crosses the r-axis at some point r_0 we then contradict the uniqueness for the Dirichlet problem, since as we have already seen (4.2) admits a monotone solution in $(0, r_0)$. If on the other hand it stays always positive we then contradict the uniqueness for the Cauchy problem: Notice that although w may increase, the transformed u function will tend to zero because of Proposition 3.5, and the previous arguments go through. We therefore conclude that G is not monotone.

This is a rather unusual way of proving that an explicitly known function is not monotone, but to do it directly the calculations are quite involved (even using MAPLE).

5. NONEXISTENCE OF SOLUTIONS

We continue our study of the semilinear elliptic Eq. (1.1) which we now write in divergence form as:

$$\nabla(\rho \nabla w) - A\rho w + \rho\, |y|^l\, |w|^{p-1}\, w = 0, \qquad \rho = e^{-|y|^2/4}. \tag{5.1}$$

We will show that for $l > 0$ and $1 < p \leqslant p_c$ the above equation admits no positive bounded solutions. To prove this, we will construct a suitable Pohozaev identity by combining three simpler identities.

Multiplying Eq. (5.1) by w and then integrating by parts over $\mathbb{R}^n$ we get the first identity:

$$\int |\nabla w|^2\, \rho + A \int w^2 \rho - \int |y|^l\, |w|^{p+1}\, \rho = 0. \tag{5.2}$$

We next multiply Eq. (5.1) by $|y|^2\, w$ and integrate by parts to get:

$$\int |y|^2\, |\nabla w|^2\, \rho - n \int |w|^2\, \rho + (A + 1/2) \int |y|^2\, w^2 \rho - \int |y|^{l+2}\, |w|^{p+1}\, \rho = 0 \tag{5.3}$$

The third identity comes from multiplying (5.1) by $(y \cdot \nabla w)$ and integrating by parts. After some calculations we find:

$$\frac{2-n}{2} \int |\nabla w|^2 \rho - \frac{An}{2} \int |w|^2 \rho + \frac{n+l}{p+1} \int |w|^{p+1} |y|^l \rho$$
$$+ \frac{1}{4} \int |y|^2 |\nabla w|^2 \rho + \frac{A}{4} \int |y|^2 w^2 \rho - \frac{1}{2(p+1)} \int |y|^{l+2} |w|^{p+1} \rho = 0. \tag{5.4}$$

To obtain the sought for Pohozaev identity we form the combination:

$$\frac{n+l}{p+1} (5.2) - \frac{A}{4(A+1/2)} (5.3) + (5.4) = 0.$$

So far we have not used the specific value of A, i.e. the fact that $A = \frac{l+2}{2(p-1)}$. If we use this value of A, we get after grouping similar terms:

$$\frac{(n-2)(p_c - p)}{2(p+1)} \int |\nabla w|^2 \rho + \frac{A(n-2)(p_c - p)\, l}{2(p+1)(p+l+1)} \int w^2 \rho$$
$$+ \frac{p-1}{4(p+l+1)} \int |y|^2 |\nabla w|^2 \rho + \frac{(p-1)\, l}{4(p+1)(p+l+1)} \int |y|^{l+2} |w|^{p+1} \rho = 0. \tag{5.5}$$

Noting that all coefficients in (5.5) are positive for the current range of p, l, we conclude that Eq. (5.1) admits no nontrivial bounded solutions.

Remark 5.1. The above calculations were made without much care about the convergence of the integrals involved. Notice however that ρ decays like a Gaussian and it is standard to show that these integrals are well defined not only for bounded solutions, but for w in a much wider class. Moreover, the result holds without any assumption of radial symmetry or positivity of the solutions.

Remark 5.2. Identity (5.5) generalizes the Pohozaev identity derived in [6] and it reduces to it, in the special case $l = 0$.

6. EXISTENCE OF INFINITE SOLUTIONS

In this section we will consider the case $p > p_c$. It is well known that when $l = 0$ and $p_c < p < p^c$ Eq. (1.1) has infinitely many solutions (cf. [3, 15]). We will show that the same happens for $l > -2$. More precisely we have:

PROPOSITION 6.1. *For $p_c < p < p^c$ and $l > -2$ there exists an infinite family of positive radial solutions of* (1.1), *say* $w_{2L}(r)$, $L = 1, 2....$ *In addition we have that*

$$w_{2L}(r)\, r^{2A} \to c_{2L}, \qquad as \quad r \to +\infty,$$

for suitable positive constants c_{2L}.

The proof of this proposition is based on the study of the intersection properties of solutions of the equation with the singular solution $U(r)$. At first we have:

LEMMA 6.2. *Let $p > \frac{n+l}{p-1}$, and $w(r)$ be a solution of Eq.* (1.1). *Then $w(r)$ cannot intersect $U(r)$ more than twice after $r_* = (pK^{p-1}/A)^{1/2}$ while staying positive.*

Proof. Set $h = w - U$ with $U(r) = Kr^{-2A}$ Then h satisfies:

$$h'' = \left(\frac{r}{2} - \frac{n-1}{r}\right) h' + (A - r^l g(r))\, h, \tag{6.1}$$

where

$$g(r) = \frac{w^p(r) - w_0^p(r)}{w(r) - U(r)}.$$

For $0 < w(r) < U(r)$ we have that $g(r) \leqslant pU^{p-1}(r)$. Therefore, for $r > r_*$ we get:

$$A - r^l g(r) \geqslant A - pU^{p-1}(r)\, r^l = A - pK^{p-1} r^{-2} > 0. \tag{6.2}$$

Suppose there is a point $r' > r_*$ at which $h(r') = 0$ and $h'(r') < 0$. It follows from (6.1) and (6.2) that for $r > r'$ we will have that $h'' < 0$ until at least $w \leqslant 0$, and the lemma is proved. ∎

We next show:

LEMMA 6.3. *Let $p > \frac{n+l}{p-1}$. For a sufficiently small, $w(r, a)$ has at most two intersection points with $U(r)$ before $w(r, a) = 0$.*

Proof. Fix an $R > r_*$ (r_* as in Lemma 6.2) and an $\varepsilon \in (0, 1/2)$. From Lemma 2.1 we know that there exists an a_0 such that for $0 < a < a_0$:

$$\left| \frac{w(r, a)}{a} - u_L(r) \right| \leqslant \varepsilon, \qquad \text{in } [0, R].$$

Choose a small enough, so that $a(u_L(R)+\varepsilon) < U(R) = KR^{-2A}$. Since u_L is an increasing function we have that:

$$0 < w(r, a) < a(u_L(R)+\varepsilon) < U(R), \qquad \text{in } [0, R].$$

In particular, $w(r, a)$ has no intersection points with U in $[0, R]$. Since $R > r_*$, it follows that $w(r, a)$ has at most two intersection points with U before $w(r, a) = 0$. ∎

We next have:

LEMMA 6.4. *Let $p_c < p < p^c$. For any integer $L \geqslant 1$, there exists a β_L such that for $\beta > \beta_L$, $w(r, \beta)$ has at least $2L+2$ intersection points with $U(r)$ before $w(r, \beta) = 0$.*

Proof. This is a consequence of Lemma 2.3 and the fact that $u_*(r)$ has infinite many intersections with $U(r)$, (cf. [16]).

Take an R big enough so that u_* has $2L+2$ intersections with U in $(0, R)$. From Lemma 2.3 it follows that the same is true for $u(r, \beta)$ for β big, say $\beta \geqslant \beta_L$. Consequently, $w(r, \beta)$ has $2L+2$ intersections with U in $(0, R\beta^{-1/2A})$. ∎

To show the existence of infinite solutions we now work as in [15]. We define the set:

$$A_{2L} = \{\alpha > 0 \mid w(r, \alpha) \text{ has at least } 2L+2 \text{ intersections with } U \text{ before } w = 0\}$$

and set $\alpha_{2L} = \inf A_{2L}$. From the lemmas proved above we have that A_{2L} is nonempty and $\alpha_{2L} > 0$. It then follows by continuity that $w(r, \alpha_{2L})$ is a positive solution of (1.1) with exactly $2L$ intersections with the singular solution U. We refer to [15] for the detailed argument.

It remains to prove the asymptotic behavior of these solutions. Since $w_{2L}(r)$ has an even number of intersections with the singular solution, we have that for large values of r:

$$w_{2L}(r) < U(r) = Kr^{-2A}.$$

We now use the change of variables already used in Lemma 3.4, $z = wr^{2A}$. Then z is bounded and satisfies equation (3.3) which we write in the form:

$$\frac{1}{\eta}(\eta z')' + \frac{z(z^{p-1} - \mu)}{r^2} = 0 \tag{6.3}$$

where $\eta \equiv r^{n-1-4A} e^{-r^2/4}$, and $\mu = K^{p-1}$ ($K > 0$ for the current range of p). Since the last term in (6.3) is of negative sign, we conclude that, for large

r, $z(r)$ cannot attain a maximum. Consequently, it is either increasing or decreasing. In either case we get that $z(r) \to c \geqslant 0$. It remains to show that $c \neq 0$.

Assuming that z decreases to zero we will reach a contradiction. Integrating (6.3) from r to ∞ we get:

$$-\eta z'(r) = \int_r^\infty \frac{z(\mu - z^{p-1})\,\eta}{s^2}\,ds < \mu \int_r^\infty \frac{z\eta}{s^2}\,ds.$$

Using the fact that z decreases and then L'Hopital's rule we have:

$$-z'(r) < \frac{\mu z(r)}{\eta} \int_r^\infty \frac{\eta}{s^2}\,ds \approx -\mu z(r)\,\frac{\eta}{\eta' r^2} \approx 2\mu\,\frac{z(r)}{r^3}.$$

It then follows that for r near infinity:

$$\left(\ln z(r) - \frac{\mu}{r^2}\right)' \geqslant 0,$$

which contradicts the assumption that z decreases to zero.

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