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Compactness and single-point blowup of positive solutions on bounded domains

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This paper is concerned with the blowup of positive solutions of the semilinear heat equation

$$u_t = \Delta u + u^p, \quad \text{on } \Omega \subset R^n, \quad 1 < p < \frac{n}{n-2},$$

with zero boundary conditions. The domain Ω is supposed to be smooth, convex and bounded. We first show that, under the assumption that the initial data are uniformly monotone near the boundary, solutions that exist on the time interval $(0, T)$ form a compact family in a suitable topology. We then derive some localisation properties of these solutions. In particular, we discuss a general criterion, independent of the initial data, which in some cases ensures single-point blowup.

1. Introduction and main results

This work is concerned with positive, blowing-up solutions of

$$\begin{aligned} u_t &= \Delta u + u^p, \quad \text{on } \Omega \subset R^n, \\ u(x, t) &= 0, \quad \text{on } x \in \partial\Omega, \end{aligned} \tag{1.1}$$

with $u(x, 0) = \varphi(x) \geqq 0$ and $1 < p < (n/(n-2))$. The domain Ω is supposed to be bounded, convex and smooth.

For $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, it is well known (cf. [**8, 24**]) that there exists a unique classical solution of (1.1) in $(0, T)$ such that either $T = +\infty$, or else $T < +\infty$ and $\| u(x, t)\|_{L^\infty} \to +\infty$ as $t \to T$. In the second case, we say that $u(x, t)$ blows up in finite time T. We say that b is a blowup point of the solution u, if there exist sequences $\{x_n\}$, and $\{t_n\}$ such that $x_n \to b$, $t_n \to T$ and $|u(x_n, t_n)| \to +\infty$.

For a large class of initial values the solution blows up. For instance, if the energy

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$E[\varphi]$ of the initial value φ is negative, that is

$$E[\varphi]=\frac{1}{2}\int|\nabla\varphi|^2-\frac{1}{p+1}\int|\varphi|^{p+1}<0,$$

then the solution of (1.1) ceases to exist in finite time, see e.g. [**17**].

A lot of work has recently been done concerning the number of blowup points, their possible location, the local properties of blowup solutions, etc. We refer to [**11, 14**] for detailed discussion and bibliography.

Let us for the moment restrict our attention to the one-dimensional case. In [**25**] Weissler proves that for symmetric initial data with one maximum, (1.1) blows up at a single point. Various authors have since proved the existence of solutions blowing up at one or two points, depending on the shape of the initial data, see e.g. [**6, 11**]. Chen and Matano [**3**] proved that, in fact, the number of blowup points is always finite and not greater than the number of local maxima of the initial data. It turned out that this result is optimal in the following sense. In [**18**] Merle has shown that given any k points situated in the interior of an interval I, there exist initial values for which the solution of (1.1) in I blows up at exactly these points. The study of the local behaviour of the solution near the blowup points is presented in [**23**].

An interesting question that has not been addressed so far, and which we intend to discuss in the present work, is the following: is there any relation between the number of oscillations in space and the blowup time? For instance, an examination of the proof in [**18**] suggests that if T_k is the blowup time of a solution blowing up at k points, then $T_k\to 0$ as k tends to infinity. Is this a general fact?

In the general n-dimensional case, it is known that the blowup points of (1.1) are contained in a compact subset of Ω, see [**6**]. In other words, the blowup set stays away from the boundary. Positive, radially symmetric initial data with one maximum yield single-point blowup (cf. [**19**]). The method of [**18**] shows that, in the unit ball, given any k numbers $0<a_1<\cdots<a_k<1$, there exists a radially symmetric blowup solution for which the blowup set is exactly equal to $\cup_{i=1}^k\{|x|=a_i\}$. Moreover, the same method shows that given any $\delta>0$ and any k point x_i, there are solutions of which the blowup set is contained in the union of the balls $B_\delta(x_i)$ centred at the x_i's with radius δ. In an earlier work, Giga and Kohn [**11**] have shown, in the case where $\Omega=R^n$, the existence of solutions blowing up at exactly an $(n-1)$-dimensional sphere. More recently, Velazquez [**22**] showed that the blowup set of (1.1) with $\Omega=R^n$ has Hausdorff dimension at most $n-1$. In view of the previous result, this estimate is sharp.

The existing examples where the blowup set is exactly known (single points, or union of lower-dimensional spheres) are based on the study of the evolution of very special (radially symmetric) initial data. In general, much more complicated structures are expected to exist (cf. [**21**]) but these are not easily to be constructed. As a matter of fact, single-point blowup is the generic behaviour (cf. [**15**]) in the sense that it is stable with respect to small perturbations of the initial data, as opposed to other blowup patterns which can be destroyed by small perturbations.

In the present work, we discuss a general criterion, independent of the initial data, which in some cases enables us to locate the blowup set.

Our first step towards the study of the localisation properties of the positive blowing-up solutions of (1.1) is the proof of a compactness property of solutions of (1.1) under the hypotheses that the initial data are uniformly monotone near the boundary. To state this last hypothesis in a more rigorous way, we need to introduce first some notation.

Let Ω_ε dentote the *tubular* neighbourhood of $\partial\Omega$ of thickness ε. Ω_ε is homeomorphic to $\partial\Omega \times (0, \varepsilon)$ and is contained in Ω. (The fact that for Ω smooth and ε sufficiently small, such a neighbourhood exists is standard; see, e.g. [**4**, Proposition 0.2].) Now, if u is a differentiable function on $\bar{\Omega}$, we can extend the normal derivative of u to the whole tubular neighbourhood by

$$\frac{\partial u}{\partial \eta}(x + t\eta(x)) = \eta \cdot \nabla u(x + t\eta(x)),$$

where $x \in \partial\Omega$, $0 \leqq t < \varepsilon$, and $\eta(x)$ is the outward normal at the point x. We then say that, for a non-negative function $\varphi \in C^1(\bar{\Omega})$,

$$\varphi \in M_\varepsilon(\Omega) \quad \text{if} \quad \frac{\partial \varphi}{\partial \eta}(x + t\eta(x)) < 0 \text{ for all } x \in \partial\Omega, \quad t \in [0, \varepsilon).$$

REMARK 1.1. Consider the one-dimensional case, with, say, $\Omega = (-1, 1)$. Then $\varphi \in M_\varepsilon$ simply means that $\varphi_x > 0$ for $x \in [-1, -1, +\varepsilon)$ and $\varphi_x < 0$ for $x \in (1 - \varepsilon, 1]$.

REMARK 1.2. The existence of functions $\varphi \in M_\varepsilon$ is rather trivial. For instance, take any $\varphi \in L^\infty$ and solve (1.1) in $(0, t_0)$ for any $t_0 > 0$. It is an easy consequence of the Hopf Maximum Principle that $u(x, t_0) \in M_\varepsilon$ for some ε; see, e.g. [**6**] or [**20**].

We denote by $u(x, t; \varphi)$ the solution of (1.1) with initial value φ. We then have the following theorem:

THEOREM 1.3. *Suppose that* $n = 1, 2$ *or, if* $n \geqq 3$ *then* $p < (n/(n-2))$. *Let* φ_α *be a family of functions from* $M_\varepsilon(\Omega)$. *Assume that* $u_\alpha = u(x, t; \varphi_\alpha)$ *solves* (1.1) *and that either it exists for all time or else it blows up at time* $T_\alpha \geqq T > 0$. *We then have*:

(i) (*boundedness*) *there holds*

$$\|u_\alpha(x, t)\|_{C^{2+\beta, 1+\beta/2}(Q_{T,\delta})} < C_\delta \quad in\ Q_{T,\delta} \equiv \Omega \times [\delta, T - \delta],$$

for any $\delta \in (0, T/2)$ *and some* $\beta \in (0, 1)$, *with the constant* C_δ *independent of* α *and depending only on* δ, T, n, p, Ω *and* (*possibly*) *on* ε;

(ii) (*compactness*) *from the family* $\{u_\alpha\}$ *we can select a sequence* $\{u_n\}$ *such that, as* $n \to +\infty$,

$$u_n(x, t) \to u_\infty(x, t) \quad in\ C^{2,1}(Q_{T,\delta}),$$

where u_∞ *satisfies* (1.1) *in* $Q_{T,\delta}$.

REMARK 1.4. Although the hypothesis that $\varphi_\alpha \in M_\varepsilon(\Omega)$ is used in the proof, the exact dependence of C_δ on ε is not clear. As a matter of fact, we suspect that this hypothesis is probably not needed, but we have not been able to remove it so far. (See also Section 4.)

Consider now the one-dimensional case, with $\Omega = I = (-1, 1)$. It is easy to see that we can find solutions which blow up at arbitrarily small or large times. For instance, if $u(x, t)$ is a solution which blows up at time T, then, by taking as initial value

$\varphi(x) = u(x, T - \varepsilon)$, the resulting solution blows up at time ε. Also, by taking as initial value φ a function which is very close to the solution of the corresponding stationary problem of (1.1), we obtain solutions with arbitrarily large blowup times.

However, if we look at one-dimensional solutions of (1.1) which have a given number of maxima for all times prior to blowup, then the blowup time can no longer be arbitrarily big. To make this more precise, let us first introduce some notation.

NOTATION 1.5. $\mathbf{T}_k = \{\sup T^{(k)} : T^{(k)}$ is the blowup time of $u_\alpha = u(x, t; \varphi_\alpha)$, $\varphi_\alpha \in M_\varepsilon(\Omega)$, where u_α is a solution of (1.1) with at least k local maxima for all $t < T^{(k)}\}$.

We then have the following theorem:

THEOREM 1.6. (i) *Suppose* $n = 1$ *and* $\Omega = I = (-1, 1)$. *There holds*:

(a) $+\infty = \mathbf{T}_1 > \mathbf{T}_2 \geqq \mathbf{T}_3 \geqq \cdots \geqq \mathbf{T}_k \geqq \mathbf{T}_{k+1} \geqq \cdots$;

(b) $\mathbf{T}_k \to 0$, *as* $k \to +\infty$.

(ii) *Suppose* $n = 1, 2$, *or if* $n \geqq 3$, *then* $p < (n/(n-2))$. *Let* $u(x, t; \varphi)$ *be a solution of* (1.1) *with* $\varphi \in M_\varepsilon(\Omega)$, *and assume that* Ω *is a ball or an ellipsoid centred at the origin. Then, there exists a constant* $\mathbf{T}$ *depending only on* Ω, n, p, *and (possibly) on* ε, *such that, if* u *blows up at time* $T \geqq \mathbf{T}$, *then* u *blows up at a single point* b. *Moreover,* b *tends to zero as* T *goes to infinity.*

REMARK 1.7. Part (i) of the above theorem is true not only for $I = (-1, 1)$ but for any bounded interval $I \subset R$, as one can easily see by using scaling. In general, the constants $\mathbf{T}_k$ depend on I, p and (possibly) on ε. The dependence on ε is 'inherited' from Theorem 1.3, since the proof of Theorem 1.6 uses the results of Theorem 1.3.

Thus, in the one-dimensional case, the blowup time of solutions with k-oscillations in space is bounded above by a constant $\mathbf{T}_k$ which is finite if $k \neq 1$. Moreover, solutions that blow up with many space oscillations should necessarily have 'small' blowup time. In contrast, if we know that a solution has a 'big' blowup time T ($T \geqq \mathbf{T}_2$), then it cannot arrive at blowup with more than one oscillation and therefore blows up at a single point.

This last property is generalised in part (ii) of Theorem 1.6 in higher dimensions. Thus, if Ω is a ball centred at the origin, solutions of (1.1) with 'big' blowup time blow up at a single point near the origin. In fact, it will follow from the proof that such a solution cannot have critical points outside a small neighbourhood of the origin, and that this neighbourhood shrinks to a point (the origin) as the blowup time goes to infinity.

A few words about notation. All integrals are meant to be taken over Ω unless otherwise specified. We denote by C_δ a generic positive constant, not necessarily the same in each occurrence, which depends on δ and possibly on other parameters as well, but is always independent of the initial values φ.

Theorem 1.3 is proved in Section 2, whereas Theorem 1.6 is proved in Section 3. In Section 4 we discuss some conjectures based on the present work.

2. A compactness property of positive blowing-up solutions

In this section, we will give the proof of Theorem 1.3. We may divide the proof in the following four steps:

(i) We first obtain a uniform L^1 bound for the solutions of (1.1) that exist in $(0, T)$.

(ii) Next, we obtain upper and lower bounds for the energy $E[u_\alpha](t)$.

(iii) Using the results of the first two steps, we bound various integral norms of the solution. All the bounds we have obtained so far are valid for times prior to blowup time.

(iv) In the final step, we use standard parabolic theory to obtain L^∞ and higher-norm bounds.

Before starting the main proof, we recall some ideas and results from [7] that we will use in the derivation of the uniform L^1 bound. Given a bounded smooth domain Ω and a direction $\gamma \in R^n$, consider the hyperplanes T_λ given by $x \cdot \gamma = \lambda$. For large positive λ, T_λ is disjoint from Ω, and as λ decreases, eventually a value $\lambda_0 = \lambda_0(\gamma, \Omega)$ is reached such that $T_{\lambda_0} \cap \partial\Omega \neq \emptyset$. For $\lambda < \lambda_0$ and near λ_0, the hyperplane cuts off a piece of Ω which is denoted by $\Sigma(\lambda, \gamma)$. Define $\Sigma'(\lambda, \gamma)$ to be the reflection of $\Sigma(\lambda, \gamma)$. As λ is further decreased, we eventually reach a value $\lambda_1 = \lambda_1(\gamma, \Omega)$ such that either $\Sigma'(\lambda, \gamma)$ is internally tangent to $\partial\Omega$ or else T_{λ_1} intersects $\partial\Omega$ somewhere orthogonally. For $\lambda \in [\lambda_1, \lambda_0)$, we call $\Sigma(\lambda, \gamma)$ a *cap* corresponding to the direction γ. For any $x \in R^n$, we denote by x^λ the reflection of x across T_λ.

PROPOSITION 2.1. *Let Ω be bounded, smooth, and u be a non-negative classical solution of* (1.1) *on* $[0, T]$ *with* $u_0(x) \in C^1(\bar{\Omega})$. *Fix a direction γ and define the caps $\Sigma(\lambda, \gamma)$ as above, for $\lambda \in [\lambda_1, \lambda_0]$. Let $\lambda \in (\lambda_1, \lambda_2)$ and*

$$u_0(x) < u_0(x^\lambda) \quad \textit{and} \quad \nabla u_0 \cdot \gamma < 0, \quad \textit{for all } x \in \Sigma(\lambda, \gamma);$$

then for all $x \in \Sigma(\lambda, \gamma)$, and $0 \leqq t \leqq T$, we have

$$u(x, t) < u(x^\lambda, t) \quad \textit{and} \quad \nabla_x u(x, t) \cdot \gamma < 0.$$

The above formulation is taken from [**20**, Proposition 6]. The proof of this proposition is based on the maximum principle and is given in [7].

We now derive the uniform L^1 bound.

PROPOSITION 2.2. *Under the assumptions of Theorem* 1.3, *we have that, given any* $\delta \in (0, T/2)$,

$$\int u_\alpha(x, t)\, dx < C_\delta, \quad t \in [0, T - \delta], \tag{2.1}$$

$$\int_0^{T-\delta} \| u_\alpha^p(\cdot, t)\|_{L^1}\, dt < C_\delta, \tag{2.2}$$

where C_δ denotes a positive constant independent of α.

Proof. Let φ_1 be the first eigenfunction of the Laplacian in Ω, that is

$$\Delta\varphi_1 = -\lambda_1\varphi_1, \quad \text{on } \Omega,$$
$$\varphi_1 = 0, \qquad \text{on } \partial\Omega.$$

We know that $\lambda_1 > 0$, $\varphi_1 > 0$. Multiplying the u equation by φ_1, we get

$$\frac{d}{dt}\int u\varphi_1 = -\lambda_1 \int u\varphi_1 + \int u^p\varphi_1. \tag{2.3}$$

Using the inequality

$$\int u^p \varphi_1 \geqq \left(\int \varphi_1\right)^{1-p} \left(\int u\varphi_1\right)^p \equiv c_0 \left(\int u\varphi_1\right)^p,$$

and setting

$$y(t) \equiv \int u\varphi_1 \, dx, \tag{2.4}$$

we have that

$$\dot{y} \geqq -\lambda_1 y + c_0 y^p.$$

Using Lemma 2.3 (see below), we get that

$$y(t) = \int u_\alpha(x, t)\varphi_1(x) \, dx < C_\delta, \quad t \in [0, T-\delta], \tag{2.5}$$

with C_δ independent of α. Integrating (2.3) in time and using (2.5), we get

$$\int_0^{T-\delta} \int u_\alpha^p(x, t)\varphi_1 \, dx \, dt < C_\delta. \tag{2.6}$$

To complete the proof, we need to remove φ_1 from (2.5) and (2.6). At this point the assumption $\varphi_\alpha \in M_\varepsilon(\Omega)$ will be used. Our argument makes use of Proposition 2.1 and is essentially the same as the one presented in [**20**, p. 113] to which we refer for more details.

Using the fact that $\varphi_\alpha \in M_\varepsilon(\Omega)$, Proposition 2.1 and arguing as in [**20**] we conclude that $u_\alpha(x, t) \in M_{\varepsilon'}(\Omega)$, uniformly in α for all $t \geqq 0$ and some $\varepsilon' \in (0, \varepsilon)$. (For instance, in the one-dimensional case we can take $\varepsilon' = \varepsilon/2$.)

Arguing once more as in [**20**], we then get that

$$\int_\Omega u_\alpha(x, t) \, dx \leqq (m+1) \int_\Omega u_\alpha(x, t) \, dx,$$

where m is a positive integer, and Ω_0 is a domain strictly contained in Ω. Both m and Ω_0 are independent of α. Using (2.5), it then follows that

$$\int_\Omega u_\alpha(x, t) \, dx \leqq \frac{m+1}{a_0} \int_\Omega u_\alpha(x, t)\varphi_1(x) \, dx \leqq \frac{m+1}{a_0} C_\delta,$$

where $a_0 \equiv \inf_{x \in \Omega_0} \varphi_1(x) > 0$. Hence, (2.1) has been proved. The proof of (2.2) is similar. $\square$

We still have to justify (2.5). We do so in the following elementary lemma.

Lemma 2.3. *Let $y(t)$ be a positive C^1 function satisfying*

$$\dot{y} \geqq -\lambda_1 y + c_0 y^p,$$

where λ_1 and c_0 are positive constants. Assume that $y(t)$ either exists for all time or it blows up ***after*** *T. We then have*

$$y(0) \leqq \left(\frac{\lambda_1}{c_0}\right)^{1/(p-1)} (1 - e^{-T\lambda_1(p-1)})^{-(1/(p-1))}. \tag{2.7}$$

Moreover, it follows from (2.7) *that*

$$y(t) \leqq C_\delta, \quad \text{for } 0 \leqq t \leqq T - \delta.$$

The constant C_δ is the same as the constant appearing in the right-hand side of (2.7) *with δ replacing T. (In particular, C_δ is independent of T.)*

Proof. Let $x = e^{\lambda_1 t} y$. Then x satisfies the differential inequality

$$\frac{d}{dt}\left(\frac{1}{x^{p-1}}\right) \leqq -c_0(p-1)e^{\lambda_1(1-p)t}.$$

We set $x(0) \equiv x_0$. Integrating the above inequality from 0 to t, we get

$$x^{p-1} \geqq \left(\frac{1}{x_0^{p-1}} - \frac{c_0}{\lambda_1}(1 - e^{-\lambda_1(p-1)t})\right)^{-1}.$$

Hence, if $x(t)$ blows up, it should blow up before the time $T_{\max}$ given by

$$\frac{1}{x_0^{p-1}} = \frac{c_0}{\lambda_1}(1 - e^{-\lambda_1(p-1)T_{\max}}). \tag{2.8}$$

We now distinguish two cases. If $y(0) = x_0$ is such that $x_0^{p-1} \leqq \lambda_1/c_0$, then (2.7) follows immediately. If $x_0^{p-1} > \lambda_1/c_0$, we can solve (2.8) for $T_{\max}$ to get

$$T_{\max} = -\frac{1}{\lambda_1(p-1)}\ln\left(1 - \frac{\lambda_1}{c_0 x_0^{p-1}}\right).$$

By hypothesis, we have that $T \leqq T_{\max}$; therefore, after some easy calculations, we obtain

$$y^{p-1}(0) = x_0^{p-1} \leqq \frac{\lambda_1}{c_0}(1 - e^{-T\lambda_1(p-1)})^{-1},$$

and (2.7) follows. □

We now proceed to obtain the upper and lower bounds for the energy $E[u_\alpha](t)$. Our first step towards the upper bound is the following lemma:

LEMMA 2.4. *Suppose that $n = 1, 2$ or if $n \geqq 3$, then $p < (n/(n-2))$. Assume that $u_\alpha = u(x, t; \varphi_\alpha)$ with $\varphi_\alpha \in M_\varepsilon(\Omega)$; either it exists for all time or else it blows up at time $T_\alpha \geqq T > 0$. Then for any $\delta \in (0, T/2)$ there exists a time $t_\alpha \in (\delta/2, \delta)$ such that*

$$\int |\nabla u_\alpha(x, t_\alpha)|^2 \, dx < C_\delta \tag{2.9}$$

Proof. Whenever there is no danger of confusion, we drop the subscript α for convenience. We have that for $t \geqq \delta/2$

$$u(t) = e^{(t-\delta/2)\Delta}u(\delta/2) + \int_{\delta/2}^{t} e^{(t-s)\Delta}u^p(s)\, ds. \tag{2.10}$$

We will use the estimate

$$\|e^{t\Delta}f\|_{L^r} \leqq Ct^{-\theta}\|f\|_{L^m}, \quad \theta = \frac{n}{2}\left(\frac{1}{m} - \frac{1}{r}\right) \geqq 0,$$

with $m=1$ and $1<r<(n/n-2)$ so that $\theta<1$. From (2.10), we get

$$\|u(t)\|_{L^r} \leqq C(t-\delta/2)^{-\theta}\|u(\delta/2)\|_{L^1} + C\int_{\delta/2}^{t} (t-s)^{-\theta}\|u^p(s)\|_{L^1}\, ds.$$

Integrating the above from $t=\delta/2$ to $t=T-\delta$, we get

$$\int_{\delta/2}^{T-\delta} \|u(t)\|_{L^r}\, dt \leqq C\int_{\delta/2}^{T-\delta} (t-\delta/2)^{-\theta}\, dt\, \|u(\delta/2)\|_{L^1} + C\int_{t=\delta/2}^{t=T-\delta}\int_{s=\delta/2}^{s=t} (t-s)^{-\theta}\|u^p(s)\|_{L^1}\, ds\, dt. \tag{2.11}$$

Using Proposition 2.2, the first term of the right-hand side is easily seen to be bounded above by some constant C_δ. To estimate the second term, we write

$$\begin{aligned}\int_{t=\delta/2}^{t=T-\delta}\int_{s=\delta/2}^{s=t} (t-s)^{-\theta}\|u^p(s)\|_{L^1}\, ds\, dt &\leqq \int_{s=\delta/2}^{s=T-\delta}\int_{t=\delta/2}^{t=T-\delta} |t-s|^{-\theta}\|u^p(s)\|_{L^1}\, dt\, ds \\ &= \int_{s=\delta/2}^{s=T-\delta} \|u^p(s)\|_{L^1}\left(\int_{t=\delta/2}^{t=T-\delta} |t-s|^{-\theta}\, dt\right) ds \leqq C_\delta,\end{aligned}$$

where we used (2.2) of Proposition 2.2.

Thus, so far, we have shown that for any $r<(n/(n-2))$ we have

$$\int_{\delta/2}^{T-\delta} \|u(t)\|_{L^r}\, dt \leqq C_\delta.$$

It follows that there exists a time

$$t_\alpha^* \in \left(\frac{\delta}{2}, \frac{2\delta}{3}\right)$$

such that

$$\|u_\alpha(t_\alpha^*)\|_{L^r} \leqq C_\delta, \quad r<\frac{n}{n-2}. \tag{2.12}$$

To conclude the proof, we now use standard local-in-time existence theory for the semilinear heat equation, following e.g. [**8**]. We first note that, since $p<(n/(n-2))$, we also have that $p>\frac{1}{2}(p-1)n$. We next pick an r such that

$$\frac{n}{n-2} > r > p > \frac{(p-1)n}{2}; \tag{2.13}$$

this is of course always possible. It follows from (2.12), (2.13) and [**8**, Theorem 1(i), (ii)] that the equation (1.1) with $u_\alpha(x, t_\alpha^*)$ as initial data has a solution in $(t_\alpha^*, t_\alpha^*+t_0)$, with t_0 independent of α. Moreover, this solution is unique ([**8**, Theorem 1(iv)] and classical (cf. [**8**, the Remark after Theorem 3]). Since it is unique, it should coincide with the solution at hand $u_\alpha(x, t)$. Thus, we can bound any norm of u_α in the time interval $(t_\alpha^*, t_\alpha^* + t_0)$ with bounds independent of α. (The bounds depend on Ω, n, p, and the constant C_δ appearing in (2.12).) In particular, by bounding the H^1 norm, we obtain that for some $t_\alpha \in (\delta/2, \delta) \cap (t_\alpha^*, t_\alpha^* + t_0)$

$$\int |\nabla u_\alpha(x, t_\alpha)|^2 \, dx < C_\delta,$$

and this completes the proof. □

Using Lemma 2.4, we now derive the upper bound for the energy E. We recall that the energy is defined by

$$E[u](t) = \frac{1}{2} \int |\nabla u|^2 - \frac{1}{p+1} \int u^{p+1}. \tag{2.14}$$

We have the following lemma:

LEMMA 2.5. *Under the assumptions of Lemma* 2.4, *we have that*

$$E[u_\alpha](t) < C_\delta, \quad \textit{for } t \in [\delta, T_\alpha]. \tag{2.15}$$

Proof. It is known that $E[u](t)$ is a monotically decreasing function of t. To see that, we just multiply the u equation by u_t and integrate by parts to get

$$\frac{d}{dt} E[u](t) = - \int u_t^2 \, dt < 0.$$

From Lemma 2.4 and the definition of E, it follows that for any α we have

$$E[u_\alpha](t_\alpha) < C_\delta, \quad \text{for some } t_\alpha \in (\delta/2, \delta),$$

and (2.15) follows from the monotonic character of E. □

We next derive a lower bound for the energy.

LEMMA 2.6. *Under the assumptions of Lemma* 2.4, *we have that*

$$E[u_\alpha](t) > -C_\delta \quad \textit{for } t \in [0, T_\alpha - \delta]. \tag{2.16}$$

Proof. We will work as in [**11**, Lemma 5.5]. We know that E is monotonically decreasing in t. Therefore, if $E[u_\alpha](t_1) \geqq 0$ for some $t_1 \in [0, T - \delta]$, then (2.16) follows at once for $0 \leqq t \leqq t_1$.

Assume now that $E[u_\alpha](t_1) < 0$ for some $t_1 < T - \delta$. Multiplying the u-equation by u and integrating by parts, we get

$$\frac{1}{2} \frac{dy}{dt} = -2E[u_\alpha](t) + \frac{p-1}{p+1} \int u_\alpha^{p+1} \, dx, \tag{2.17}$$

with

$$y(t) \equiv \int u_\alpha^2 \, dx.$$

Using the monotonicity of E and Holder's inequality, we get

$$\frac{dy}{dt} \geqq -4E[u_\alpha](t_1) + cy^{(p+1)/2}, \quad t \geqq t_1. \tag{2.18}$$

Since we have assumed $E[u_\alpha](t_1) < 0$, this inequality forces finite-time blowup, and the blowup time T^* is estimated above by

$$T^* - t_1 \leqq \int_0^\infty \frac{dy}{4|E_\alpha(t_1)| + cy^{(p+1)/2}} = C|E(t_1)|^{-\gamma},$$

with $\gamma = (p-1)/(p+1)$ and $C = C(n, p)$ depending only on n and p.

Since $T_\alpha \leqq T^*$, we also have that

$$-E[u_\alpha](t_1) = |E_\alpha(t_1)| \leqq \left(\frac{C}{T^* - t_1}\right)^{1/\gamma} \leqq \left(\frac{C}{T_\alpha - t_1}\right)^{1/\gamma} \leqq \left(\frac{C}{\delta}\right)^{1/\gamma}.$$

Thus, (2.16) has been proved in all cases. □

Using the results we have obtained so far, we can now bound various integral norms of the solution.

Proposition 2.7. *Suppose that $n = 1, 2$ or if $n \geqq 3$, then $p < (n/(n-2))$. Assume that $u_\alpha(x, t)$ either exists for all time or else it blows up at time $T_\alpha \geqq T$. We then have:*

$$\int_\delta^{T_\alpha - \delta} \int u_{\alpha t}^2 \, dx \, dt < C_\delta, \tag{2.19}$$

$$\int u_\alpha^2 \, dx < C_\delta \quad \textit{for all } t \in [\delta, T - \delta], \tag{2.20}$$

$$\int_\delta^{T-\delta} \left(\int u_\alpha^{p+1} \, dx \right)^2 dt < C_\delta, \tag{2.21}$$

$$\int_\delta^{T-\delta} \left(\int |\nabla u_\alpha|^2 \, dx \right)^2 dt < C_\delta. \tag{2.22}$$

Proof. We will argue as in [**10**, Proposition 2.2] (see also [**11**, Proposition 3.1]). Whenever there is no danger of confusion we drop the subscript α for convenience.

We first recall the identity

$$\frac{d}{dt} E[u_\alpha](t) = -\int u_{\alpha t}^2 \, dx. \tag{2.23}$$

Integrating (2.23) in time, we get

$$\int_\delta^{T_\alpha - \delta} \int u_{\alpha t}^2 \, dx \, dt = E[u_\alpha](\delta) - E[u_\alpha](T_\alpha - \delta),$$

and (2.19) follows since, by Lemmas 2.5 and 2.6, $E[u_\alpha]$ is uniformly bounded from above and from below.

In order to prove (2.20) we set (so that we match the notation of [**10**])

$$g(t) = \left(\int u^2 \, dx \right)^{\frac{1}{2}}.$$

We now look for an L^∞ estimate for $g(t)$. Working as in the derivation of (2.18), we obtain

$$2E[u](\delta) + g\dot{g} \geqq cg^{p+1},$$

whence

$$\text{either } g(t) \leqq 1 \quad \text{or } cg^p \leqq \dot{g} + 2E[u](\delta).$$

In particular,

$$\int_\delta^{T-\delta} g^{2p}(s)\, ds \leqq T - 2\delta + c^{-2} \int_\delta^{T-\delta} (2|\dot{g}|^2 + 8E^2[u](\delta))\, ds. \tag{2.24}$$

Also, from the definition of $g(t)$, we have that

$$\dot{g} \leqq g^{-1} \int uu_t \, dx \leqq \left(\int u_t^2 \, dx \right)^{\frac{1}{2}}.$$

Using (2.19), we get

$$\int_\delta^{T-\delta} |\dot{g}|^2 \, ds \leqq \int_\delta^{T-\delta} \int u_t^2 \, dx\, dt \leqq C_\delta.$$

Thus, the right-hand side of (2.24) is bounded by some constant C_δ independent of α. To conclude the proof of (2.20), we finally use the Sobolev inequality

$$\|g\|_{L^\infty} \leqq C(\|\dot{g}\|_{L^2} + \|g\|_{L^2})^a \|g\|_{L^{2p}}^{1-a}, \quad a = \frac{1}{p+1},$$

valid for functions $g(t)$ in $(\delta, T - \delta)$.

The proof of the remaining two bounds is simpler. In order to prove (2.21), we just integrate (2.23) in time and use standard inequalities. Finally, (2.22) follows from the definition of the energy (2.14) and the previous bounds. We omit further details. □

We are now ready to give the proof of Theorem 1.3.

Proof of Theorem 1.3. We need to show that

$$\|u_\alpha(x, t)\|_{C^{2+\beta, 1+\beta/2}(Q_{T,\delta})} \leqq C_\delta. \tag{2.25}$$

We will use standard parabolic regularity. Using (2.21) and Schwarz's inequality, we get

$$\int_\delta^{T-\delta} \int u_\alpha^{p+1} \, dx\, dt < C_\delta \tag{2.26}$$

for some constant C_δ independent of α. Now, u_α solves the equation

$$u_{\alpha t} - \Delta u_\alpha = u_\alpha^p, \quad \text{in } Q_{T,\delta} \equiv \Omega \times [\delta, T - \delta],$$

with Dirichlet boundary conditions, and we have from (2.25) that u_α^p is bounded in $L^{(p+1)/n}(Q_{T,\delta})$, the bound being independent of α. Using L^q regularity theory [**16**, Chap. 4, p. 335] we conclude that $u_{\alpha t}$, ∇u_α, $\nabla^2 u_\alpha$ are in $L^{(p+1)/p}(Q_{T,\delta})$. Therefore, by Sobolev inequality [**16**, Chap. 2, Lemma 3.3, p. 80] we deduce that u_α^p is bounded in $L^r(Q_{T,\delta})$ for some $r > (p+1)/n$. By bootstrapping, we eventually get that u_α is Holder continuous, so that Shauders' estimates apply [**5**, Chap. 3, Theorem 5, p. 64]. We finally conclude that u_α, $u_{\alpha t}$, ∇u_α, $\nabla^2 u_\alpha$ are Holder-continuous with, say, exponent β in $Q_{T,\delta}$, uniformly with respect to α. Thus, (2.25) has been proved. Part (ii) follows easily from the uniform bound (2.25) and the Arzela-Ascoli Theorem. □

3. Dependence of the blowup set by the blowup time

In this section, we will give the proof of Theorem 1.6. As before, we assume throughout the section that the initial data are in $M_\varepsilon(\Omega)$ for some ε.

We begin by studying the one-dimensional problem in the interval $I = (-1, 1)$:

$$u_t = u_{xx} + u^p, \quad x \in I, \quad u(\pm 1, t) = 0. \tag{3.1}$$

We would like to show that the number of oscillations in space is bounded above by a function of time only, independent of the initial values. To make this more precise, we first introduce some notation.

Notation 3.1. We denote by $n(\alpha, t)$ the number of critical points of $u_\alpha(x, t)$ (or, equivalently, the number of zeros of $u_{\alpha x}(x, t)$) at time t.

We then have the following proposition:

Proposition 3.2. *There exists a finite, positive, nonincreasing, integer-valued function $N(t)$ such that*

$$n(\alpha, t) \leqq N(t), \quad \textit{for } t \in (0, T). \tag{3.2}$$

Remark 3.3. We note that this is not true for an unbounded domain. Indeed, let $I = R$ and suppose that $u(x, t)$ is a solution of (3.1) with, say, k maxima at time t_0. By rescaling, we have that $u_\lambda = \lambda^{2/(p-1)} u(\lambda x, \lambda^2 t)$ is also a solution of (3.1) with k maxima at time $t_{0\lambda} \equiv t_0/\lambda^2$, and $t_{0\lambda}$ can be made arbitrarily large by taking λ suitably small.

Proof of Proposition 3.2. It is a consequence of our compactness Theorem 1.3 and the results of Angenent [**1**] about the zeroset of parabolic equations. The fact that $N(t)$ is positive and integer-valued is obvious.

We now prove (3.2). We will prove it by contradiction. Suppose that (3.2) is not true. Then, for some time t_0, and for every $m = 1, 2, \ldots$ there exists a solution of (1.1) $u_m(x, t)$, having at least m critical points at $t = t_0$. Let $v_m \equiv u_{mx}$. Then v_m satisfies the equation

$$v_{mt} = m_{mxx} + p u_m^{p-1} v_m, \quad \text{in } [-1, 1] \times (0, T), \tag{3.3}$$

with $p u_m^{p-1} \in L^\infty_{\text{loc}}(0, T; L^\infty(-1, 1))$. From the Maximum Principle, we also have that $v_m(-1, t) > 0$ and $v_m(1, t) < 0$ for all $t \in (0, T)$. We conclude from [**1**, Theorem D] that $v_m(x, t)$ has at least m zeros for any $t \in (0, t_0)$.

Using our Theorem 1.3, we can extract a subsequence, still denoted by $\{u_m\}$, such that

$$u_m(x,t)\to u_\infty(x,t), \quad \text{as } m\uparrow\infty, \quad \text{in } C^2((-1,1)\times(0,T)). \tag{3.4}$$

It follows from (3.4) and the properties of u_m, that $v_\infty(x,t)\equiv u_{\infty x}(x,t)$ has either infinitely many zeros, or else (at least) a multiple zero for any $t\in(0,t_0)$.

From (3.3) and (3.4), it follows that v_∞ solves the equation (3.3) with u_∞ replacing u_m. Using again [**1**, Theorem D] we get that u_∞ cannot have infinitely many zeros for any t. It is also an easy consequence of [**1**, Theorem D], that $v_\infty(x,t)$ cannot have a multiple zero for infinitely many times (cf. e.g. [**3**, Lemma 2.4(ii)]). Thus, we have reached a contradiction, and this completes the proof. Finally, by the results of [**1**], $N(t)$ is finite and nonincreasing. □

Let us recall the definition of $\mathbf{T}_k$ from the Introduction.

DEFINITION 3.4. $\mathbf{T}_k=\{\sup\{T^{(k)}: T^{(k)}$ is the blowup time of $u(x,t;\varphi_\alpha)$, where $u(x,t;\varphi_\alpha)$ is a solution of (1.1) with at least k local maxima for all $t<T^{(k)}\}$.

We then have the next lemma:

LEMMA 3.5. *There holds*:
(i) $\mathbf{T}_k\geqq\mathbf{T}_{k+1}$, *for* $k=1,2,\dots$;
(ii) $\mathbf{T}_1=+\infty$;
(iii) $\mathbf{T}_k\to 0$, *as* $k\to+\infty$.

Proof. (i) This is an immediate consequence of Definition 3.4 and the fact that the number of maxima does not increase with time (see [**1**]).

(ii) We need to show that there are solutions of (1.1) with one maximum which blow up at arbitrarily large times. Let $\varphi_0(x)$ be the (unique) positive solution of

$$u_{xx}+u^p=0, \quad x\in I,$$
$$u(-1)=u(1)=0.$$

Using φ_0 as the initial data of (1.1), the solution $u(x,t;\varphi_0)=\varphi_0(x)$ exists for all times. Moreover, it is well known that φ_0 is radially symmetric with a single maximum (cf. e.g. [**7**]). Consider now the family of initial data $\varphi_\epsilon=(1+\epsilon)\varphi_0$. An easy calculation shows that

$$E[\varphi_\epsilon]=\left(\frac{(1+\epsilon)^2}{2}-\frac{(1+\epsilon)^{p+1}}{p+1}\right)\int\varphi_0^{p+1}<0.$$

Thus, $u(x,t;\varphi_\epsilon)$ blows up at finite time, say $T_\epsilon<+\infty$. Since the blowup time depends continuously on the initial data (see [**18**, Proposition 2.1]), we conclude that T_ϵ tends to infinity as $\epsilon\to 0$.

(ii) This is a consequence of Proposition 3.2. We will prove it by contradiction. Suppose this is not true. Then for any $k=1,2,\dots$ there exists a solution $u_k(x,t;\varphi_\alpha)$ having at least k maxima, and a T_0 such that $T^{(k)}>T_0$; here $T^{(k)}$ is the blowup time of u_k. It follows from Proposition 3.2 that for $t<T_0$ and for all $k=1,2,\dots$ we have

$$k\leqq n(\alpha,t)\leqq n(\alpha,T_0)\leqq N(T_0),$$

which is a contradiction since $N(T_0)$ is a fixed number. □

We now consider the general n-dimensional case. It is well known that solutions of (1.1) which exist for all time, as time tends to infinity, approach zero, or infinity, or a positive solution of the stationary problem

$$\begin{aligned} \Delta u + u^p = 0, &\quad \text{on } \Omega, \\ u = 0, &\quad \text{on } \partial\Omega. \end{aligned} \tag{3.5}$$

Let us assume for simplicity that (3.5) admits a unique positive solution. This is true, for instance, in the case where Ω is a ball or an ellipsoid. In particular, it is true in the one-dimensional case. We then have the next proposition:

PROPOSITION 3.6. *Let $u_n(x, t)$ be a sequence of positive solutions of* (1.1), *such that $\varphi_n \in M_\varepsilon(\Omega)$ and u_n blows up at $T_n < +\infty$. We assume that $T_n \to +\infty$ as $n \to +\infty$. Consider a sequence of times t_n such that $t_n \to +\infty$ and $T_n - t_n \to +\infty$ as $n \to +\infty$. We then have that*

$$u_n(x, t_n) \to w(x), \quad \text{in } C^2(\Omega), \quad \text{as } n \to +\infty; \tag{3.6}$$

here $w(x)$ is the unique positive solution of (3.5).

Proof. The proof will be given in various steps which we label for convenience.

(a) From Proposition 2.7, we have that

$$\int_0^{T_n - \delta} \| u_{nt}(\cdot, t) \|_{L^2}^2 \, dt < C_\delta, \tag{3.7}$$

with C_δ independent of T_n.

(b) There exists $\tau_n \in [t_n/4, t_n/3]$ such that as $n \to +\infty$

$$\int_{\tau_n}^{\tau_n + 1} \| u_{nt}(\cdot, t) \|_{L^2}^2 \, dt \to 0. \tag{3.8}$$

Indeed, if (3.8) were not true, then for all $\tau_n \in [t_n/4, t_n/3]$ there would exist a $c_0 > 0$ such that

$$\int_{\tau_n}^{\tau_n + 1} \| u_{nt}(\cdot, t) \|_{L^2}^2 \, dt \geqq c_0 > 0.$$

But then

$$\int_{t_n/4}^{t_n/3} \| u_{nt}(\cdot, t) \|_{L^2}^2 \, dt \geqq \frac{t_n c_0}{12} \to +\infty, \quad \text{as } n \to +\infty,$$

which is a contradiction. Thus, (3.8) has been proved. We may assume that $\tau_{n+1} \geqq \tau_n \geqq \ldots$ by passing to a subsequence if necessary.

(c) We set

$$v_n(x, t) = u_n(x, t + \tau_n).$$

We then have from (3.8) that

$$\lim_{n \to +\infty} \int_0^1 \| v_{nt}(\cdot, t) \|_{L^2}^2 \, dt = 0. \tag{3.9}$$

(d) By our compactness Theorem 1.3, we have that there exists a subsequence such that

$$v_n(x, t) \to v_\infty(x, t), \quad \text{in } C^{2,1}(\Omega \times (0, 1)). \tag{3.10}$$

(e) It follows that v_∞ is a solution of

$$v_{\infty t} = \Delta v_\infty + v_\infty^p, \quad \text{in } \Omega \times (0, 1),$$

with Dirichlet boundary conditions. Moreover, from (3.9) we also have that v_∞ is independent of time. Thus, v_∞ is a non-negative solution of (3.5).

(f) Here we will show that $v_\infty(x) \not\equiv 0$. We claim that if $u(x, t_0; \varphi) \leqq \epsilon_0$, for some ϵ_0 sufficiently small and some $t_0 \geqq 0$, then u exists for all time. Let us accept this for the moment and continue. Suppose that $v_\infty \equiv 0$. Then, from (3.10), we get that, given any $\epsilon_0 > 0$, there exists an n such that

$$u_n(x, t + \tau_n) < \epsilon_0.$$

It then follows that $u_n(x, t)$ exists for all time. But this is a contradiction since, by hypothesis, u_n blows up at $T_n < +\infty$. Consequently $v_\infty \not\equiv 0$.

We still have to substantiate our claim. We do so by following the arguments of [**20**, Proposition 2] where a stronger statement is proved: pick a domain $\hat{\Omega}$ which strictly contains Ω. Let $\hat{\varphi}_1$ denote the first eigenfunction of $-\Delta$ in $\hat{\Omega}$, with the normalisation $\|\hat{\varphi}_1\|_{L^\infty} = 1$. An easy calculation shows that for ϵ sufficiently small, say $\epsilon < \epsilon_1$, $\epsilon\hat{\varphi}_1$ is a supersolution of (1.1). Set

$$\epsilon_2 = \min_{x \in \bar{\Omega}} (\epsilon_1 \hat{\varphi}_1).$$

Then, if

$$\|u(\cdot, t_0; \varphi)\|_{L^\infty} \leqq \epsilon_2,$$

for some time t_0, then

$$\|u(\cdot, t; \varphi)\|_{L^\infty} \leqq \epsilon_1,$$

for all times $t \geqq t_0$. This proves our claim.

Since we have assumed that $w(x)$ is the unique positive solution of (3.5), we have shown that

$$u_n(x, \tau_n) \to w(x), \quad \text{in } C^2(\Omega), \quad \text{as } n \to +\infty.$$

(g) Working similarly as in the steps (c)–(f), we obtain that there exists $s_n \in [2t_n/3, 3t_n/4]$ such that

$$u_n(x, s_n) \to w(x), \quad \text{in } C^2(\Omega), \quad \text{as } n \to +\infty.$$

(h) From the identity

$$E[u_n](s_n) - E[u_n](\tau_n) = -\int_{\tau_n}^{s_n} \|u_{nt}(\cdot, t)\|_{L^2}^2 \, dt$$

and the fact that

$$\lim_{n \to +\infty} (E[u_n](s_n) - E[u_n](\tau_n)) = E[w] - E[w] = 0,$$

it follows in particular that

$$\lim_{n\to+\infty}\int_{t_n-1}^{t_n+1}\|u_{nt}(\cdot,t)\|_{L^2}^2\,dt=0.$$

(i) Working once more as in (c)–(f), we finally obtain that

$$u_n(x,t_n)\to w(x),\quad\text{in } C^2(\Omega),\quad\text{as } n\to+\infty,$$

as claimed. □

We now have the following consequences of the above proposition:

LEMMA 3.7. *Let $u(x,t;\varphi)$, with $\varphi\in M_\varepsilon(I)$, denote a positive solution of (3.1). Then there exists a constant $\mathbf{T}$ such that if u blows up at $T\geqq\mathbf{T}$ then the blowup set of u consists of a single point, say b. Moreover, b tends to zero as T tends to infinity.*

Proof. This is a consequence of Proposition 3.6 and the Maximum Principle (in the form of Proposition 2.1). First we prove the single-point blowup statement. Suppose this were not true. Then there would exist solutions u_n of (3.1) such that u_n blows up at time $T_n\geqq n$ and the blowup set contains at least two points.

From Proposition 3.6, it follows that we have

$$u_n\left(x,\frac{n}{2}\right)\to w(x)\quad\text{in } C^2(\Omega)\quad\text{as } n\to+\infty. \tag{3.11}$$

Since $w(x)$ is radially symmetric and decreasing, we have that $w_{xx}(x)<0$ for all $x\in I$, therefore it follows from (3.11) that, for n large enough,

$$u_{nxx}\left(x,\frac{n}{2}\right)<0,\quad x\in I.$$

Thus, $u_n(x,n/2)$ has a single maximum, and by the results of [1] it will keep having a single maximum for all $t\in(n/2,T_n)$. It follows that $u_n(x,t)$ blows up at a single point, contradicting our original assumption.

We still have to show that the blowup point b tends to zero as T increases. Suppose this were not true. Consider the sequence u_n of solutions of (3.1) having the same properties as before. We then would have that there exists an $\epsilon_0>0$ such that for all n large enough, u_n blows up at a point $b_n\in(-1,-\epsilon_0)\cup(\epsilon_0,1)$. From (3.11), we have that

$$u_{nx}\left(x,\frac{n}{2}\right)>0,\quad x\in(-1,-\epsilon_0)$$

and (using the notation of Proposition 2.1)

$$u_n\left(x,\frac{n}{2}\right)<u_n\left(x^\lambda,\frac{n}{2}\right),\quad x,\lambda\in(-1,-\epsilon_0).$$

It then follows from Proposition 2.1 that

$$u_{nx}(x,t)>0,\quad\text{for } x\in(-1,-\epsilon_0)\quad\text{and}\quad t\in\left(\frac{n}{2},T_n\right). \tag{3.12}$$

Thus, if u_n blows up at $b_n \in (-1, -\epsilon_0)$. (3.12) would force u_n to blow up (at least) on the whole interval $(b_n, -\epsilon_0)$, which is impossible. Similarly, b_n cannot be in $(\epsilon_0, 1)$. We have thus reached a contradiction and this completes the proof. □

REMARK 3.8. The same proof works in higher dimensions in the case where Ω has suitable symmetries. In particular, the same proof works for Theorem 1.6(ii).

To complete the proof of Theorem 1.6, we still have to show the finiteness of the $\mathbf{T}_k$'s. This is an immediate consequence of Lemma 3.7:

LEMMA 3.9. *Let* $\mathbf{T}_k$ *be as defined in Definition* 3.4. *Then*

$$\mathbf{T}_k < +\infty \quad \textit{for } k \geqq 2.$$

Proof. If this were not true, then there would exist solutions of (3.1) with more than one maximum, and arbitrarily large blowup times. But it follows from the proof of Lemma 3.7 that this is impossible. It also follows that we can take $\mathbf{T} = \mathbf{T}_2$.

4. Discussion and some open problems

About the hypothesis on the initial values

An unpleasant aspect of our analysis is the hypothesis about the monotonicity property of the initial values. This hypothesis has been used only in the derivation of the uniform L^1 estimate (Proposition 2.2). We wonder whether this is an essential assumption, or simply a technical one which can somehow be removed.

It is interesting to note that for the *linear* heat equation, given any family of non-negative initial values φ from $C^1(\bar{\Omega})$, the corresponding family of solutions $u(x, t; \varphi)$ is uniformly monotone near the boundary for any fixed $t > 0$. More precisely, the following result has been proved by the authors:

PROPOSITION 4.1. *Let* $\Omega \in R^n$ *be a bounded, strictly convex domain, and* $u = u(x, t; \varphi)$, *with* $\varphi \geqq 0$, *and let* $\varphi(x) \in C^1(\bar{\Omega})$ *denote the solution of the* ***linear*** *heat equation on* Ω, *with Dirichlet boundary conditions and* φ *as initial value. Then, for any* $t > 0$, *we have that*

$$u(x, t) \in M_{\delta(t)}(\Omega),$$

where $\delta(t)$ *is a nonincreasing function which depends only on* t *and* Ω. *Moreover*, $\delta(t) = O(t^{\frac{1}{2}})$ *as* $t \to 0$.

The proof of this result is based on a careful study of the Green's function near the boundary, and the Maximum Principle (in the form of Proposition 2.1). Unfortunately, the proof does not work for the nonlinear problem.

Thus, an interesting question is to know whether a similar property is true for solutions of the nonlinear problem (1.1). If the answer is positive, the assumption on the initial values can be removed.

Space dimension one

It is known (cf. [**2**, **12**, **13**]) that for problem (1.1), different blowup patterns are possible depending on whether one maximum reaches the blowup time, or two (or in general k) maxima coalesce at exactly the blowup time. In the first case, we say that we have a *simple* blowup point, whereas in the second we have a *double* blowup point (or, in general, a blowup point of *multiplicity* k).

Consider initial data for which the solution of (3.1) blows up at $T^{(2)}$ with at least two maxima for all times $t < T^{(2)}$. By our Theorem 1.6, we have that $T^{(2)} \leq \mathbf{T}_2 < +\infty$. An interesting question is to characterise the solutions for which $T^{(2)}$ is equal to $\mathbf{T}_2$.

Given the fact that all solutions that blow up after $\mathbf{T}_2$ blow up necessarily at a single (and simple) point, one could conjecture that solutions that blow up at exactly $\mathbf{T}_2$ are having two maxima for times $t < \mathbf{T}_2$ and these two maxima coalesce at time $t = \mathbf{T}_2$. The same remark applies for all $\mathbf{T}_k$, so that finally we have the following conjecture:

CONJECTURE 4.2. *Consider the family of solutions which blow up with at least k maxima for all times prior to blow up. Among these solutions there is one which maximises the blowup time (that is, for which* $T^{(k)} = \mathbf{T}_k$*), and having the property that it blows up at a single point with multiplicity k.*

Higher dimensions

Here we expect that some version of Theorem 1.6(ii) holds true for a general bounded smooth and convex domain. More precisely:

CONJECTURE 4.3. *Let u be a solution of* (1.1) *which blows up at time T. Then, for all* $t > 0$, *there exists a domain* $\Omega_t \subset \Omega$ *such that u has no critical points in* $\Omega \backslash \Omega_t$ *and* $\Omega_t \to \{x_0\}$ *as* $t \to +\infty$. *Moreover, there exists a constant* $\mathbf{T}$ *depending only on* Ω, n, p, *such that if* $T > \mathbf{T}$ *then u blows up at a single point.*

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