

SEMICLASSICAL WIGNER FUNCTION AND GEOMETRICAL OPTICS*

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Abstract. We consider the problem of high-frequency asymptotics for the time-dependent one-dimensional Schrödinger equation with rapidly oscillating initial data. This problem is commonly studied via the WKB method. An alternative method is based on the limit Wigner measure. This approach recovers geometrical optics, but, like the WKB method, it fails at caustics. To remedy this deficiency we employ the semiclassical Wigner function which is a formal asymptotic approximation of the scaled Wigner function but also a regularization of the limit Wigner measure. We obtain Airy-type asymptotics for the semiclassical Wigner function. This representation is shown to be exact in the context of concrete examples. In these examples we compute both the semiclassical and the limit Wigner function, as well as the amplitude of the wave field near a fold or a cusp caustic, which evolve naturally from suitable initial data.

Key words. geometrical optics, Wigner equation, phase space, semiclassical limit, caustics

AMS subject classifications. 34E05, 34E13, 34E20, 58K55, 78A05, 81S30, 81Q20

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1. Introduction. We consider the Cauchy problem for the time-dependent one-dimensional Schrödinger equation with fast space-time scales,

$$(1.1) \quad i\epsilon\psi_t^\epsilon = -\frac{\epsilon^2}{2}\psi_{xx}^\epsilon + V(x)\psi^\epsilon(x, t), \quad x \in \mathbb{R},$$

and highly oscillatory initial data:

$$(1.2) \quad \psi^\epsilon(x, 0) = \psi_0^\epsilon(x) = A_0(x) \exp(iS_0(x)/\epsilon).$$

The parameter ϵ appears in both the equation and the initial data, and it is considered to be a small parameter. We are interested in the high-frequency limit of (1.1), (1.2), that is, in the limit of ψ^ϵ as ϵ tends to zero.

Apart from quantum mechanics, (1.1), (1.2) arises in many contexts in classical wave propagation as the paraxial approximation of forward propagating waves [FLAT]. Thus, it is of practical importance for computing wave intensities in many applied fields such as radioengineering [FOC], laser optics [TAP1], underwater acoustics [TAP2], the investigation of light and sound propagation in turbulent atmosphere [TAT1], and seismic wave propagation in the earth's crust [SF], to mention but a few. In these cases, the potential V is explicitly related to the refraction index of the propagating medium.

Problems of high-frequency waves such as (1.1), (1.2) have traditionally been studied via the method of geometrical optics (see, e.g., [BLP], [BB], [KO1]). We briefly review this method for the problem at hand. The starting point is the WKB

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ansatz according to which the asymptotic solution of (1.1), (1.2) as $\epsilon \rightarrow 0$ has the form

$$(1.3) \quad \psi^\epsilon(x, t) = A(x, t) \exp\left(iS(x, t)/\epsilon\right).$$

Substituting (1.3) into (1.1) and retaining terms of order $O(1)$ in ϵ , we obtain the following system for the phase $S(x, t)$ and the principal amplitude $A(x, t)$:

$$(1.4) \quad S_t + (S_x)^2/2 + V = 0,$$

$$(1.5) \quad 2A_t + 2A_x S_x + A S_{xx} = 0.$$

Introducing the Hamiltonian

$$(1.6) \quad H(x, k) = k^2/2 + V(x),$$

where $k \in \mathbb{R}$ is the momentum, we rewrite (1.4) in the standard form of the Hamilton–Jacobi (eikonal) equation

$$(1.7) \quad S_t + H(x, \partial_x S) = 0.$$

The phase function $S(x, t)$ is constructed by the method of characteristics (rays) as follows. First, we solve the ODEs

$$(1.8) \quad \frac{d\bar{x}}{dt} = H_k = \bar{k}, \quad \frac{d\bar{k}}{dt} = -H_x = -V'(\bar{x}),$$

with initial conditions

$$(1.9) \quad \bar{x}(0) = r, \quad \bar{k}(0) = S'_0(r),$$

to obtain the characteristics $(\bar{x}(t; r), \bar{k}(t; r))$. Then the phase $S(\bar{x}(t; r), t)$ is obtained by integrating the equation

$$(1.10) \quad \frac{dS}{dt} = S_t + S_x H_k = S_t + \bar{k} \cdot \bar{k} = -H + \bar{k}^2 = \frac{\bar{k}^2}{2} - V(\bar{x}),$$

with initial condition $S(x, 0) = S_0(r)$, along the rays.

On the other hand, applying the divergence theorem in a ray tube for the transport equation (1.5) (see [BLP]), we obtain the following formula for the principal amplitude:

$$(1.11) \quad A(\bar{x}(t; r), t) = A_0(r) J^{-\frac{1}{2}}(t; r),$$

where

$$(1.12) \quad J(t; r) = \frac{\partial \bar{x}(t; r)}{\partial r}$$

is the Jacobian of the ray transformation $r \rightarrow \bar{x}(t; r)$.

Once $S(x, t)$ and $A(x, t)$ have been found, a first approximation of ψ^ϵ is given by (1.3). This amounts to a quick review of the WKB method. Since the nonlinear equation (1.4) for the phase function $S(x, t)$ does not in general have global in time solutions, the WKB method fails on caustics where it predicts infinite wave amplitudes. From the mathematical point of view, formation of caustics corresponds to

the multivaluedness of the phase function $S(x, t)$ due to rays crossing. On the other hand, formation of caustics is a common situation in most wave problems as a result of multipath propagation from localized sources. Indeed, even in the simplest oceanographic models and geophysical structures (see, e.g., [TC], [CMP]), various types of caustics occur, depending upon the position of the source and the stratification of the wave velocities.

Assuming that the multivalued phase function is known, uniform asymptotic formulas for the wave field near the caustics have been constructed using boundary layer techniques [BAK], [BUK] as well as phase-space techniques, and notably Lagrangian integrals [LU], [KR], [DUI], [GS], [KO2], and the method of canonical operator [MF], [VA]. Given the practical importance of the problem, a number of numerical techniques have been proposed in order to compute the multivalued phase functions; see, e.g., [BEN], [EFO], [ER], [RU] and the references therein.

A relatively new phase-space technique for studying oscillatory solutions of dispersive wave equations and the homogenization of energy density $\eta^\epsilon(x, t) = |\psi^\epsilon(x, t)|^2$ is based on the use of the Wigner transform (see, e.g., [GM], [GMMP], [BCKP], [JL]). The Wigner transform of the wave function ψ^ϵ converges, as ϵ goes to zero, to the so-called limit Wigner measure $f^0(x, k, t)$ which solves a Liouville equation in phase space [LP], [GMMP]. Since the Liouville equation is easily solved, we can obtain f^0 for all $t \geq 0$ and from this the “homogenized” energy density

$$(1.13) \quad \eta^0(x, t) = \int_{-\infty}^{\infty} f^0(x, k, t) dk.$$

In particular, for the problem (1.1), (1.2), the initial Wigner measure $f^0(x, k, t = 0)$ is a Dirac mass, and it remains so for all times. Then, for a single-phase wave field, it follows from (1.13) that $\eta^0(x, t) = A^2(x, t)$, the amplitude A given by the solution (1.11) of the transport equation (1.5). For multiphase fields a similar result arising from linear superposition holds. For example, assuming a two-phase field (as in the vicinity of a fold caustic) where WKB method predicts $\psi^\epsilon(x, t) = A_+(x, t)e^{iS^+(x, t)/\epsilon} + A_-(x, t)e^{iS^-(x, t)/\epsilon}$, (1.13) yields

$$(1.14) \quad \eta^0(x, t) = A_+^2(x, t) + A_-^2(x, t),$$

A_+, A_- being the solutions of the transport equation (1.5) with $S = S^+$ or $S = S^-$, respectively (cf. [SMM, Thm. 4.2]). Thus, this approach recovers the WKB solution as long as (x, t) lies in the illuminated zones. However, the Liouville equation fails to predict the correct multiphase solution either on the caustics, where the limit Wigner measure f^0 is not well defined, or in the shadow zones, where this measure vanishes. To elucidate this fact we explicitly compute the Wigner function f^ϵ on and away from the caustics for specific multiphase examples. Thus, for a fold, we find that away from the caustic and in the illuminated zone, f^ϵ converges weakly to a sum of two Dirac masses, and one can recover (1.14). On the other hand, on the fold itself, $\epsilon^{1/3}f^\epsilon$ converges to a single Dirac mass, and it follows that $\eta^\epsilon(x, t) = O(\epsilon^{-1/3})$ as ϵ goes to zero. Clearly, the last two estimates (on the fold) cannot be obtained from the limit Wigner function.

In order to capture the correct field on the caustics and in the shadows, one should work with the scaled Wigner function f^ϵ instead of f^0 . The function f^ϵ satisfies the full Wigner equation (see (2.7) below). This is an infinitely singular (as ϵ goes to zero), transport-dispersive integrodifferential equation, and thus many analytical as well as numerical difficulties are anticipated.

It is the main objective of this work to construct formal asymptotic approximations \tilde{f}^ϵ of f^ϵ which are valid even on caustics. To this end the starting point is the WKB solution. Assuming that $0 \leq t \leq T$, with $T < t_c$, t_c being the first time when a caustic appears, we show that the Wigner transform $W^\epsilon(x, k, t)$ of the WKB solution is a formal asymptotic solution of the full Wigner equation. Using stationary phase techniques at each fixed time t , we obtain Airy-type asymptotics of W^ϵ , which we denote by $\tilde{W}^\epsilon(x, k, t)$ and call the *semiclassical Wigner function*. This semiclassical object can be thought of as an approximation of $f^\epsilon(x, k, t)$. Although \tilde{W}^ϵ is a phase-space quantity it is not defined on the caustics, since for $t = t_c$ the WKB solution itself fails.

To overcome this difficulty we first consider the special case $V(x) = ax^2 + bx + c$, ($a, b, c \in \mathbb{R}$). The key observation here is that the evolution of the initial semiclassical Wigner function $\tilde{W}^\epsilon(x, k, t = 0)$ under the Hamiltonian flow is an approximation of $\tilde{W}^\epsilon(x, k, t)$ for $0 \leq t \leq T$, and in addition it is well defined even for $t = t_c$. This motivates the definition of \tilde{f}^ϵ as the evolution of the initial semiclassical Wigner function $\tilde{W}^\epsilon(x, k, t = 0)$ under the Hamiltonian flow. A similar construction applies to the case of more general potentials.

The quantity \tilde{f}^ϵ thus defined converges to the limit Wigner (Dirac) mass away from caustics, and it satisfies the analogue of (1.13), that is,

$$\eta^\epsilon(x, t) = \int_{-\infty}^{\infty} \tilde{f}^\epsilon(x, k, t) dk + o(1) \quad \text{as } \epsilon \rightarrow 0.$$

Furthermore, it conveys the appropriate local scales even on caustics so that one can compute the correct amplitude there.

In our simple examples \tilde{f}^ϵ coincides with f^ϵ , and we find analytic expressions for both the scaled and the limit Wigner function, as well as for the amplitude of the wave field near a fold or a cusp caustic, which evolve naturally from suitable initial data. These expressions reveal the structure of the oscillations of the Wigner function as well as the way the scaled Wigner converges towards the limit Wigner (Dirac) mass.

The paper is organized as follows:

- Section 2 is devoted to the basics of the Wigner transform and in the construction of the semiclassical (Airy) Wigner function as a local asymptotic approximation of the Wigner transform of a WKB function.

- In section 3 we show that the Wigner transform of a single-phase WKB solution is a formal approximation of the solution of the full Wigner equation. Based on that, we construct asymptotic approximations of the solution of the full Wigner equation.

- In section 4 we present in detail two specific examples revealing the structure of the solution of the Wigner equation near fold and cusp caustics generated by the initial data.

- In the final section we present a review of the main points of this work, along with some concluding remarks.

2. The Wigner transform.

2.1. The Wigner equation. For any smooth function $\psi(x)$ rapidly decaying at infinity with respect to x , say, $\psi \in \mathcal{S}(\mathbb{R})$, the Wigner transform of ψ is defined by (see, e.g., [LP], [PR], [WIG])

$$(2.1) \quad W(x, k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-ik\sigma) \psi\left(x + \frac{\sigma}{2}\right) \bar{\psi}\left(x - \frac{\sigma}{2}\right) d\sigma,$$

where $\bar{\psi}$ denotes the complex conjugate of ψ . This is a function defined in phase space, and it has many important properties. First, it is real and its k -integral is the modulus square of ψ ,

$$(2.2) \quad \int_{\mathbb{R}} W(x, k) dk = |\psi(x)|^2.$$

Thus, if ψ is a wave function, we may think of $W(x, k)$ as wave number resolved energy density. This is not quite precise because $W(x, k)$ is not in general positive (except when ψ is a Gaussian function; see, e.g., [FOL, p. 62]), but it always becomes positive in the high frequency limit. Second, the energy flux is expressed through $W(x, k)$ by

$$(2.3) \quad \mathcal{F} = \frac{1}{2i} \left(\psi(x) \overline{\psi'(x)} - \overline{\psi(x)} \psi'(x) \right) = \int_{\mathbb{R}} kW(x, k) dk.$$

Third, given a wave function of the form $\psi^\epsilon(x) = A(x) \exp(iS(x)/\epsilon)$, its *scaled Wigner transform*

$$(2.4) \quad W^\epsilon(x, k) = \frac{1}{\epsilon} W \left(x, \frac{k}{\epsilon} \right)$$

has, as a generalized function, the weak limit [LP], [PR]

$$(2.5) \quad W^\epsilon(x, k) \rightarrow |A(x)|^2 \delta(k - S'(x)), \quad \epsilon \rightarrow 0.$$

This suggests that the correct scaling for the high-frequency limit for the problem (1.1) is

$$(2.6) \quad W^\epsilon(x, k, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-ik\sigma) \psi^\epsilon \left(x + \frac{\epsilon\sigma}{2}, t \right) \overline{\psi^\epsilon \left(x - \frac{\epsilon\sigma}{2}, t \right)} d\sigma.$$

Using the Wigner transform (2.6) to the Schrödinger equation (1.1), we obtain the integrodifferential *Wigner equation* [LP], [PR], [GMMP]

$$(2.7) \quad L^\epsilon[f^\epsilon] = f_t^\epsilon(x, k, t) + kf_x^\epsilon(x, k, t) + \mathcal{Z}_\epsilon f^\epsilon(x, k, t) = 0,$$

where the operator \mathcal{Z}_ϵ is defined by the convolution with respect to the momentum k ,

$$(2.8) \quad \mathcal{Z}_\epsilon f(x, k, t) = f(x, k, t) *_k \frac{i}{2\pi\epsilon} \int_{-\infty}^{\infty} \exp(-iky) \left(V \left(x + \frac{\epsilon}{2}y \right) - V \left(x - \frac{\epsilon}{2}y \right) \right) dy.$$

Assuming that the potential V is smooth enough, we can expand $V(x \pm \frac{\epsilon}{2}y)$ into Taylor series and rewrite (2.7) as a singular equation of infinite order (cf. [TAT2]):

$$(2.9) \quad f_t^\epsilon + kf_x^\epsilon - V'(x)f_k^\epsilon = \sum_{m=1}^{\infty} \alpha_m \epsilon^{2m} V^{(2m+1)}(x) \partial_k^{2m+1} f^\epsilon,$$

where $\alpha_m = \frac{(-1)^m}{2^{2m}(2m+1)!}$, $m = 0, 1, \dots$, and $V^{(2m+1)}(x) = \frac{d^{2m+1}V(x)}{dx^{2m+1}}$. Formally speaking, (2.9) is a transport-dispersive equation, since the left-hand side contains a standard transport (Liouville) operator, while the right-hand side is a dispersive operator

of infinite order. Since, in general, (2.9) involves infinitely many derivatives of the unknown function multiplied by powers of the small parameter ϵ , it can be thought of as a singular perturbation problem of infinite order.

The initial condition for (2.7) (or (2.9)) is

$$f_0^\epsilon(x, k, t = 0) = W_0^\epsilon(x, k, t = 0),$$

where W_0^ϵ is the Wigner transform of the initial datum of (1.2).

In the formal limit $\epsilon = 0$, the right-hand side of (2.9) is zero, and (2.9) reduces to the *limit Wigner equation*

$$(2.10) \quad f_t^0(x, k, t) + kf_x^0(x, k, t) - V'(x)f_k^0(x, k, t) = 0,$$

which is a simple transport equation in the phase space \mathbb{R}_{xk}^2 .

Note finally that the limit Wigner transform of the WKB solution (1.3) is given, according to (2.5), by

$$(2.11a) \quad f^0(x, k, t) = A^2(x)\delta(k - S'(x, t)).$$

It is also easily verified, by substituting (2.11a) into (2.10) and using the Hamilton–Jacobi and transport (1.4) and (1.5), respectively (cf. [PR]), that f^0 is a solution of the Liouville equation (2.10), with initial condition

$$(2.11b) \quad f_0^0(x, k) = A_0^2(x)\delta(k - S'_0(x)),$$

the limit Wigner transform of the initial datum (1.2). A direct solution of (2.10) will be given in section 3.3 below.

2.2. Asymptotics of the Wigner transform of a WKB function. In this section we will derive the asymptotics of the Wigner transform of a WKB function. Our analysis is motivated by the work of Berry [BER1] and Berry and Balazs [BEBA], who derived Airy-type expansions of the Wigner function corresponding to stationary solutions of (1.1) in terms of geometric characteristics of the closed fixed-energy curve (assumed globally convex) in phase space.

The (scaled) Wigner transform of a WKB function

$$\psi^\epsilon(x) = A(x) \exp(iS(x)/\epsilon)$$

is given by the oscillatory integral

$$(2.12) \quad W^\epsilon(x, k) = \frac{1}{\pi\epsilon} \int_{-\infty}^{\infty} D(\sigma, x) \exp\left(\frac{i}{\epsilon}F(\sigma, x, k)\right) d\sigma,$$

where

$$(2.13) \quad D(\sigma, x) := A(x + \sigma)A(x - \sigma)$$

and

$$(2.14) \quad F(\sigma, x, k) := S(x + \sigma) - S(x - \sigma) - 2k\sigma.$$

Asymptotic expansions of oscillatory integrals as in (2.12) are, in general, constructed applying the method of stationary phase; cf. [BOR], [BH]. According to this

method the main contribution to W^ϵ comes from the stationary points of the Wigner phase F , that is, the points σ which are roots of the equation

$$(2.15) \quad F'_\sigma(\sigma, x, k) = S'(x + \sigma) + S'(x - \sigma) - 2k = 0.$$

Clearly, the roots of (2.15) appear in symmetric pairs $\pm\sigma_0(x, k)$. The nature of these roots depends on the position of the point (x, k) relative to the Lagrangian manifold

$$\Lambda = \{(x, k) \mid k = S'(x)\}$$

associated with the WKB function $\psi^\epsilon(x)$ (cf. [ARN1], [ARN2]).

We therefore consider the following three cases.

Case 1. $\pm\sigma_0 \neq 0$ are simple stationary points. Let us assume that at some point (x, k) there corresponds two simple stationary points $\pm\sigma_0(x, k) \neq 0$. In particular we have that $F''_\sigma(\sigma_0, x, k) = S''(x + \sigma_0) - S''(x - \sigma_0) \neq 0$. Then the standard stationary phase formula applies, and it leads to the semiclassical approximation

$$(2.16) \quad W^\epsilon(x, k) \approx 4(2\pi\epsilon)^{-\frac{1}{2}} \frac{D(\sigma_0, x)}{\sqrt{|F''_{\sigma\sigma}(\sigma_0, x, k)|}} \times \cos\left(\frac{1}{\epsilon}F(\sigma_0, x, k) + \frac{\pi}{4}\text{sign}(F''_{\sigma\sigma}(\sigma_0, x, k))\right).$$

Although the first term of (2.16) is of order $O(\epsilon^{-1/2})$, it is multiplied by a highly oscillatory factor $(\cos(F/\epsilon))$, and it tends weakly to zero.

To explain this, for fixed x , let us consider the point $P = (x, k_0)$ in the concave side of Λ ; see Figure 1. Then $\sigma_0(x, k_0)$ is uniquely determined by the endpoints Q and R of the chord having P as its midpoint (Berry's chord construction). Clearly, for k close to k_0 , $F(\sigma, x, k)$ has two simple stationary points $\pm\sigma_0(x, k) \neq 0$. In particular (2.16) is valid with $\sigma_0 = \sigma_0(x, k)$. A Taylor expansion about k_0 yields

$$F(\sigma_0(x, k), x, k) = F(\sigma_0(x, k_0), x, k_0) - 2\sigma_0(x, k_0)(k - k_0) + O((k - k_0)^2).$$

Using the weak limit $\epsilon^{-m} \exp(ix/\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$, $m \geq 0$, we see that for k close to k_0 , the right-hand side of (2.16) goes to zero as $\epsilon \rightarrow 0$.

We note that as P approaches $T = (x, S'(x)) \in \Lambda$, then both Q and R approach T , and hence σ_0 goes to zero. In this case the two simple stationary points $\pm\sigma_0$ coalesce. We therefore consider the next case.

Case 2. $\pm\sigma_0 = 0$. It must be emphasized $\sigma_0 = 0$ is a stationary point only in the case where $(x, k) \in \Lambda$, that is, $k = S'(x)$; cf. (2.15). Moreover, $\sigma_0 = 0$ is a multiple stationary point since $F'_\sigma(0, x, S'(x)) = F''_{\sigma\sigma}(0, x, S'(x)) = 0$, and (2.16) does not apply in this case. This multiple stationary point results from the coalescence of the simple stationary points $\pm\sigma_0(x, k) \neq 0$ (of Case 1) which move towards zero as (x, k) approaches Λ , as shown in Figure 1.

Under the assumption that we have a double stationary point, that is,

$$F'''_{\sigma\sigma\sigma}(0, x, S'(x)) = 2S'''(x) \neq 0,$$

a uniform approximation of the integral (2.12) can be derived by employing the method of Chester, Friedman, and Ursell [CFU]; see Appendix A. This method concerns one-parameter phase integrals for which two stationary points coalesce to a double point as a control parameter, say α , approaches zero.

In order to rigorously apply this method, we first fix x and then take $\alpha = k - S'(x)$ as the small parameter which controls the distance between the stationary points $\pm\sigma_0$;

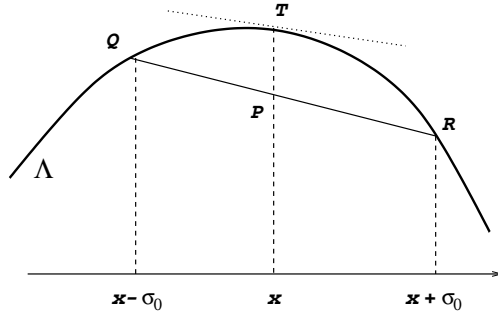


FIG. 1. Coalescence mechanism for a double stationary point $\sigma_0 = 0$.

notice that $\sigma_0 \rightarrow 0$ as $\alpha \rightarrow 0$. Then the uniform approximation formula (A.3) of Appendix A (with $B_0 = 0$ due to the symmetry of the stationary points) leads to the expansion (semiclassical Wigner function)

$$(2.17a) \quad \begin{aligned} W^\epsilon(x, k) &\approx D\left(-2^{\frac{1}{2}} \left(\frac{k-S'(x)}{S'''(x)}\right)^{\frac{1}{2}}, x\right) \\ &\times \frac{2^{\frac{2}{3}}}{\epsilon^{\frac{2}{3}}} \left(\frac{2}{|S'''(x)|}\right)^{\frac{1}{3}} Ai\left(-\frac{2}{\epsilon^{\frac{2}{3}}} \frac{k-S'(x)}{(S'''(x))^{\frac{1}{3}}}\right) + O(\epsilon^{\frac{4}{3}}). \end{aligned}$$

This expansion can be also formally constructed as follows. We expand both D and F in Taylor series about $\sigma = 0$,

$$D(\sigma, x) = D(0, x) + O(\sigma), \quad F(\sigma, x, k) = -2(k - S'(x))\sigma + \frac{1}{3}S'''(x)\sigma^3 + O(\sigma^5).$$

Discarding the term $O(\sigma)$ from D and $O(\sigma^5)$ from F and recalling the integral representation of the Airy function (see Appendix B), it follows from (2.12) that

$$(2.17b) \quad W^\epsilon(x, k) \approx D(0, x) \frac{2^{\frac{2}{3}}}{\epsilon^{\frac{2}{3}}} \left(\frac{2}{|S'''(x)|}\right)^{\frac{1}{3}} Ai\left(-\frac{2}{\epsilon^{\frac{2}{3}}} \frac{k - S'(x)}{(S'''(x))^{\frac{1}{3}}}\right).$$

The apparent difference between the formal expansion (2.17b) and the rigorous one (2.17a) lies in the approximation of the amplitude D , which in the latter case is simply $D(0, x) = A^2(x)$, while in the former it has a more precise form. Nevertheless, near the manifold the two formulas are in good agreement. We denote by $\widetilde{W}^\epsilon(x, k)$ the approximation

$$(2.18) \quad \widetilde{W}^\epsilon(x, k) := \frac{2}{\epsilon^{\frac{2}{3}}} \frac{A^2(x)}{|S'''(x)|^{\frac{1}{3}}} Ai\left(-\frac{2}{\epsilon^{\frac{2}{3}}} \frac{k - S'(x)}{(S'''(x))^{\frac{1}{3}}}\right).$$

Taking into account the weak limit

$$(2.19) \quad (1/\epsilon)Ai(y/\epsilon) \rightarrow \delta(y), \quad \epsilon \rightarrow 0,$$

it follows easily that

$$(2.20) \quad \widetilde{W}^\epsilon(x, k) \rightarrow A^2(x)\delta(k - S'(x)), \quad \epsilon \rightarrow 0.$$

Thus, in the classical limit ($\epsilon = 0$), the right-hand side of (2.17b) recovers the limit Wigner (Dirac) function; cf. (2.5). In addition, the projection formula (2.2) is satisfied

if instead of W^ϵ we use the right-hand side of (2.17b). Indeed, using the fact that $\int_{\mathbb{R}} Ai(y)dy = 1$, we easily compute that

$$(2.21) \quad \int_{\mathbb{R}} \widetilde{W}^\epsilon(x, k)dk = \frac{2}{\epsilon^{\frac{2}{3}}} \frac{A^2(x)}{|S'''(x)|^{\frac{1}{3}}} \int_{\mathbb{R}} Ai\left(-\frac{2}{\epsilon^{\frac{2}{3}}} \frac{k - S'(x)}{(S'''(x))^{\frac{1}{3}}}\right) dk = A^2(x),$$

thus recovering the correct amplitude.

The approximations (2.17a) and (2.17b) are not valid near degenerate points of Λ , that is, near points where $S'''(x) = 0$. If we assume that near such a point $S^{(2N+1)}(x) \neq 0$ is the first nonzero odd derivative, then, by keeping more terms in the Taylor expansion of F about $\sigma = 0$, we can formally write the following formula:

$$(2.22) \quad W^\epsilon(x, k) \approx \frac{1}{2\pi} A^2(x) \int_{-\infty}^{+\infty} \exp\left\{i\left[-(k - S'(x))\sigma + \sum_{m=1}^N \frac{S^{(2m+1)}(x)}{(2m+1)!} \frac{\epsilon^{2m}}{2^{2m}} \sigma^{2m+1}\right]\right\} d\sigma.$$

We note that this approximation satisfies the analogue of (2.20) or (2.21). Indeed, by setting $\epsilon = 0$ in (2.22) we recover the Dirac mass, whereas by interchanging the s and k integration we recover the amplitude via the projection formula (2.2).

Case 3. $\pm\sigma_0 \neq 0$ are multiple stationary points. We note that the previous two cases are only possible in the case where Λ is globally convex or concave. For arbitrary Λ , however, there will be points (x_0, k_0) to which there correspond nonzero multiple stationary points $\pm\sigma_0$. Similarly to Case 1, W^ϵ is highly oscillatory in the neighborhood of these points, and it tends (weakly) to zero as ϵ goes to zero.

To see that, we consider the simple (nongeneric) case where $\pm\sigma_0(x_0, k_0) \neq 0$ are double stationary points, that is, $F'_\sigma(\pm\sigma_0, x_0, k_0) = F''_{\sigma\sigma}(\pm\sigma_0, x_0, k_0) = 0$ and

$$F'''_{\sigma\sigma\sigma}(\pm\sigma_0, x_0, k_0) = S'''(x_0 + \sigma_0) + S'''(x_0 - \sigma_0) \neq 0.$$

Such a case is shown geometrically in Figure 2. We consider the points (x_0, k) , where k is close to k_0 . Expanding in Taylor series about $\pm\sigma_0$ both $D(\sigma, x_0, k)$ and $F(\sigma, x_0, k)$ and working as in Case 2, we obtain after some calculations that

$$(2.23) \quad W^\epsilon(x_0, k) \approx W_+^\epsilon(x_0, k) + W_-^\epsilon(x_0, k),$$

with

$$(2.24) \quad \begin{aligned} W_\pm^\epsilon(x_0, k) &= \exp\left(-\frac{i}{\epsilon}\left(\pm 2\sigma_0(k - k_0) + F(\sigma_0, x_0, k_0)\right)\right) D(\pm\sigma_0, x_0) \\ &\times \frac{2^{\frac{2}{3}}}{\epsilon^{\frac{2}{3}}} \left(\frac{2}{|F'''_{\sigma\sigma\sigma}(\pm\sigma_0, x_0, k_0)|}\right)^{\frac{1}{3}} Ai\left(-\frac{2}{\epsilon^{\frac{2}{3}}} \frac{k - k_0}{(F'''_{\sigma\sigma\sigma}(\pm\sigma_0, x_0, k_0))^{\frac{1}{3}}}\right). \end{aligned}$$

Although the expression in the second line of (2.24) tends to the Dirac mass $\delta(k - k_0)$, the right-hand side of (2.24) tends to zero, due to the rapid oscillations of the exponential factor $\exp(-\frac{i}{\epsilon}(\pm 2\sigma_0(k - k_0) + F(\sigma_0, x_0, k_0)))$. Note that this could not happen in Case 2 considered above, since then $\sigma_0 = 0$ and $F(0, x_0, k_0) = 0$, and the oscillations in front of the Airy function disappear. Thus, only the asymptotics coming from the zero stationary points (Case 2) can give nontrivial contributions as ϵ tends to zero.

Conclusion. The essential asymptotic contribution to W^ϵ comes from the points near the Lagrangian manifold Λ , where W^ϵ is generically (i.e., when $S'''(x) \neq 0$)

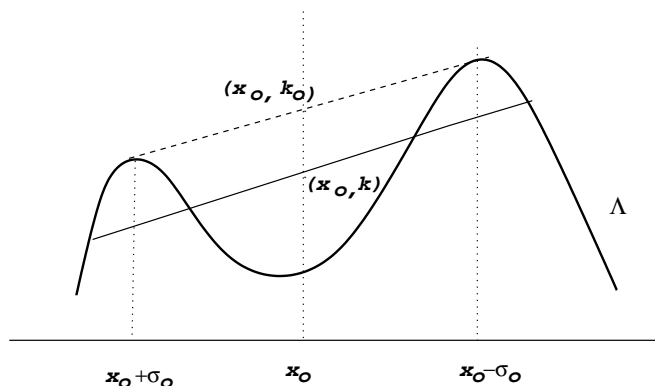


FIG. 2. Coalescence mechanism for a double stationary point $\sigma_0 \neq 0$.

approximated by the standard Airy function, that is,

$$(2.25) \quad W^\epsilon(x, k) \approx \widetilde{W}^\epsilon(x, k) = \frac{2}{\epsilon^{\frac{2}{3}} |S'''(x)|^{\frac{1}{3}}} A^2(x) Ai\left(-\frac{2}{\epsilon^{\frac{2}{3}}} \frac{k - S'(x)}{|S'''(x)|^{\frac{1}{3}}}\right).$$

We have seen that as $\epsilon \rightarrow 0$, the right-hand side of (2.25) converges to the correct limit Wigner (Dirac) function; cf. (2.20). Moreover, its integral with respect to k is equal to $A^2(x)$ (cf. (2.21)) in agreement with the projection formula (2.2).

We recall that to obtain approximation (2.25) we have “thrown away” highly oscillatory terms that tend weakly to zero. As a consequence, this approximation is of *local* nature; that is, it is valid for k close to $S'(x)$. Indeed, for k outside a boundary layer of thickness of order $O(\epsilon^{\frac{2}{3}} |S'''(x)|^{\frac{1}{3}})$ about Λ , both \widetilde{W}^ϵ and W^ϵ tend weakly to zero due to rapid oscillations. Clearly, as we approach a degenerate point, the thickness of this boundary layer shrinks to zero.

By means of (2.19) we may think of (2.25) as being valid even at degenerate points, in the sense that if $S'''(x) = 0$ we may replace the right-hand side of (2.25) by $A^2(x)\delta(k - S'(x))$. Of course, this is only a matter of convenience, since at degenerate points the Airy asymptotics are lost, and we simply recover the limit Wigner.

Our formal argument suggests that near degenerate points one should use generalized Airy approximations; cf. (2.22).

3. Single-phase optics. In this section we consider the single-phase case. That is, our analysis is restricted in the time interval $0 \leq t \leq T$, with $T < t_c$, where t_c is the first time a caustic appears.

3.1. The WKB solution. We recall the Wigner equation (cf. (2.7))

$$(3.1) \quad L^\epsilon[f^\epsilon] = \partial_t f^\epsilon(x, k, t) + k\partial_x f^\epsilon(x, k, t) + \mathcal{Z}_\epsilon f^\epsilon(x, k, t) = 0.$$

When looking for approximate solutions of (3.1), the most natural candidate is the Wigner transform of the WKB solution

$$\psi_\epsilon(x, t) = A(x, t)e^{iS(x, t)/\epsilon},$$

where S and A satisfy the eikonal and transport equations (1.4) and (1.5), respectively. The Wigner function of ψ_ϵ is given by

$$(3.2) \quad W^\epsilon(x, k, t) = \frac{1}{\pi\epsilon} \int e^{\frac{i}{\epsilon}(S(x+\sigma, t) - S(x-\sigma, t) - 2k\sigma)} A(x + \sigma, t)A(x - \sigma, t)d\sigma.$$

We need to clarify in what sense W^ϵ is an approximate solution of the Wigner equation. To this end we will plug the expression for W^ϵ into (3.1), and we will compute the left-hand side.

To simplify the calculations we introduce the notation

$$\begin{aligned}\Phi(\sigma; x, t) &:= S(x + \sigma, t) - S(x - \sigma, t), & F(\sigma; x, k, t) &:= \Phi - 2k\sigma, \\ D(\sigma; x, t) &:= A(x + \sigma, t)A(x - \sigma, t), & Q(\sigma; x) &:= V(x + \sigma) - V(x - \sigma).\end{aligned}$$

The following relations are immediate:

$$\begin{aligned}\Phi_x &= F_x = S_x(x + \sigma, t) - S_x(x - \sigma, t), \\ \Phi_\sigma &= S_x(x + \sigma, t) + S_x(x - \sigma, t), \\ F_\sigma &= \Phi_\sigma - 2k, \\ \Phi_{x\sigma} &= S_{xx}(x + \sigma, t) + S_{xx}(x - \sigma, t), \\ S_x(x \pm \sigma, t) &= \frac{1}{2}(\Phi_\sigma \pm \Phi_x).\end{aligned}$$

We then rewrite (3.1) as

$$(3.3) \quad W^\epsilon = \frac{1}{\pi\epsilon} \int e^{\frac{i}{\epsilon}F} D d\sigma.$$

Using the eikonal and transport equations we derive some helpful relations involving Φ , D , and Q . The relevant calculations are lengthy but straightforward, and we therefore omit the details. We write the eikonal equation at the points $x + \sigma$ and $x - \sigma$, multiply by $\psi_\epsilon(x - \sigma)$ and $\bar{\psi}_\epsilon(x + \sigma)$, respectively, take the Fourier transform, and then subtract the resulting expressions to obtain

$$(3.4) \quad (\partial_t + k\partial_x)F = (\partial_t + k\partial_x)\Phi = -\frac{1}{2}(F_\sigma\Phi_x + 2Q).$$

In a similar manner, from the transport equation, we obtain

$$(3.5) \quad (\partial_t + k\partial_x)D = -\frac{1}{2}(F_\sigma D_x + \Phi_x D_\sigma + D\Phi_{x\sigma}).$$

We finally compute the convolution term in (3.1):

$$(3.6) \quad \mathcal{Z}_\epsilon W^\epsilon = \frac{i}{\pi\epsilon^2} \int e^{\frac{i}{\epsilon}F} DQ d\sigma.$$

Plugging (3.3) in (3.1) and using (3.6), we get

$$\begin{aligned}L^\epsilon[W^\epsilon] &= (\partial_t + k\partial_x)W^\epsilon + \mathcal{Z}_\epsilon W^\epsilon(x, k, t) \\ &= \frac{1}{\pi\epsilon} \int e^{\frac{i}{\epsilon}F} \left(\frac{i}{\epsilon} D(\partial_t + k\partial_x)\Phi + \frac{i}{\epsilon} DQ + (\partial_t + k\partial_x)D \right) d\sigma.\end{aligned}$$

Using (3.4) and (3.5) we have

$$\begin{aligned}L^\epsilon[W^\epsilon] &= -\frac{1}{2\pi\epsilon} \int \left(e^{\frac{i}{\epsilon}F} \right)_\sigma \Phi_x D d\sigma - \frac{1}{2\pi\epsilon} \int e^{\frac{i}{\epsilon}F} \left(F_\sigma D_x + \Phi_x D_\sigma + D\Phi_{x\sigma} \right) d\sigma \\ &= -\frac{1}{2\pi\epsilon} \int \left(e^{\frac{i}{\epsilon}F} \Phi_x D \right)_\sigma d\sigma - \frac{1}{2\pi\epsilon} \int e^{\frac{i}{\epsilon}F} F_\sigma D_x d\sigma \\ &= -\frac{1}{2\pi\epsilon} \int e^{\frac{i}{\epsilon}F} F_\sigma D_x d\sigma.\end{aligned}$$

Integrating the last integral by parts we conclude that

$$(3.7) \quad L^\epsilon[W^\epsilon] = \frac{i}{2\pi} \int e^{iF} D_{x\sigma} d\sigma := \frac{i}{2\pi} I_\epsilon(x, k, t).$$

Hence, W^ϵ as defined in (3.2) is an approximate solution of the Wigner equation as $\epsilon \rightarrow 0$ to the extent that the integral I_ϵ in (3.7) approaches zero as $\epsilon \rightarrow 0$.

The oscillatory integral in (3.7) is similar to the oscillatory integral in section 2.2; cf. (2.12). A similar analysis can be made for I_ϵ ; however, since there is an extra multiplicative ϵ in I_ϵ (compared to W^ϵ), a rough estimate can be obtained as follows. According to the stationary phase method, the main contribution to I_ϵ as $\epsilon \rightarrow 0$ at the point (x, k, t) will come from the points σ for which $F_\sigma(\sigma; x, k, t) = 0$. Let σ_0 be such a point. Assume that $F_\sigma(\sigma_0) = F_{\sigma\sigma}(\sigma_0) = \dots = F_\sigma^{(n-1)}(\sigma_0) = 0$, whereas $F_\sigma^{(n)}(\sigma_0) \neq 0$. Taking the Taylor expansion of F and $D_{x\sigma}$ about $\sigma = \sigma_0$, we see that the contribution from σ_0 in I_ϵ is

$$I_{\sigma_0, \epsilon} \approx e^{iF(\sigma_0)} D_{x\sigma}(\sigma_0; x, t) \int e^{i \frac{F_\sigma^{(n)}(\sigma_0)}{n!} (\sigma - \sigma_0)^n} d\sigma.$$

The last integral is easily estimated from above so that

$$|I_{\sigma_0, \epsilon}| \leq C\epsilon^{1/n}.$$

Assuming, for instance, that $A(x, t)$ has compact support and that $S(x, t)$ is analytic, it is standard to see that there are only a finite number of such points σ_0 , and hence

$$I_\epsilon \approx \sum_{\sigma_0} I_{\sigma_0, \epsilon} = O(\epsilon^{1/n})$$

for some positive integer n . Hence, we conclude that $I_\epsilon(x, k, t)$ tends pointwise to zero, and consequently W^ϵ defined by (3.2) is a formal approximation of the solution of (3.1).

In fact, one can prove that f^ϵ and W^ϵ are also close in the L^2 -sense, that is,

$$(3.8) \quad \|f^\epsilon - W^\epsilon\|_{L^2} \leq C_T t \epsilon^{1/2}, \quad 0 \leq t \leq T.$$

For a proof see Appendix C.

By the results of section 2.2 and away from degenerate points (i.e., $S_{xxx}(x, t) \neq 0$), $f^\epsilon(x, k, t)$ is approximated by \widetilde{W}^ϵ , that is,

$$(3.9) \quad f^\epsilon(x, k, t) \approx \widetilde{W}^\epsilon(x, k, t) = \frac{2}{\epsilon^{\frac{2}{3}}} \frac{A^2(x, t)}{|S_{xxx}(x, t)|^{\frac{1}{3}}} Ai\left(-\frac{2}{\epsilon^{\frac{2}{3}}} \frac{k - S_x(x, t)}{(S_{xxx}(x, t))^{\frac{1}{3}}}\right).$$

3.2. Phase-space dynamics. In order to understand how \widetilde{W}^ϵ evolves with time, we must compute how the quantities $k - S_x$ and S_{xxx} evolve with time.

In the single-phase optics, the initial *Lagrangian manifold* Λ_0 associated with the initial data (1.2),

$$\Lambda_0 = \{(q, p) : p = S'_0(q)\},$$

evolves with the Hamiltonian flow $g_t : (q, p) \mapsto (x(t; q, p), k(t; q, p))$ to the Lagrangian manifold [ARN1]

$$\Lambda_t = \{(x, k) : k = S_x(x, t)\},$$

where $S(x, t)$ is the (single-valued) solution of the eikonal equation (1.7).

The flow g_t is described by the Hamiltonian system

$$\frac{dx}{dt} = k, \quad \frac{dk}{dt} = -V'(x),$$

with initial conditions $x(0) = q, k(0) = p$; here (q, p) is any initial point in phase space \mathbb{R}_{xk}^2 .

We will derive an ODE for the quantity $k(t; q, p) - S_x(x(t; q, p), t)$. Clearly, for $t = 0$ we have

$$k(0; q, p) - S_x(x(0; q, p), 0) = p - S'_0(q).$$

Differentiating along the bicharacteristic

$$\Gamma_{q,p} = \{(x, k) \mid x = x(t; q, p), k = k(t; q, p), t \geq 0\},$$

we have

$$(3.10) \quad \begin{aligned} \frac{d}{dt}(k - S_x(x, t)) &= \frac{dk}{dt} - (\partial_t + k\partial_x - V'(x)\partial_k)S_x(x, t) \\ &= \frac{dk}{dt} - (\partial_t + k\partial_x)S_x(x, t) = -V'(x) - S_{xt}(x, t) - kS_{xx}(x, t). \end{aligned}$$

We use now the eikonal equation to replace the first two terms of the right-hand side of (3.10). Taking the ∂_x -derivative of the eikonal equation we get

$$S_{xt}(x, t) + S_x(x, t)S_{xx}(x, t) + V'(x) = 0.$$

Then from (3.10) we obtain

$$\frac{d}{dt}(k - S_x(x, t)) = -S_{xx}(x, t)(k - S_x(x, t)).$$

Integrating this equation along the bicharacteristic $\Gamma_{q,p}$, we find

$$(3.11) \quad p - S'_0(q) = \exp\left(\int_0^t S_{xx}(x(\tau; q, p), \tau)d\tau\right)(k - S_x(x, t)).$$

We next recall the formula [KR], [SMI]

$$(3.12) \quad J(t; q) = \exp\left(\int_0^t S_{xx}(x(t; q, S'_0(q)), \tau)d\tau\right);$$

for the Jacobian

$$J(t; q) = \frac{\partial \bar{x}}{\partial q}(t; q),$$

along the ray $\bar{x}(t; q) = x(t; q, S'_0(q))$; see AC in Figure 3.

Note that in single-phase optics, for a given (\bar{x}, t) , $q = q(\bar{x}, t)$ is uniquely defined. For p close to $S'_0(q)$ and fixed q , we have that

$$(3.13a) \quad S_{xx}(x(\tau; q, p), \tau) = S_{xx}(\bar{x}(\tau; q), \tau) + O(|x(\tau; q, p) - \bar{x}(\tau; q)|).$$

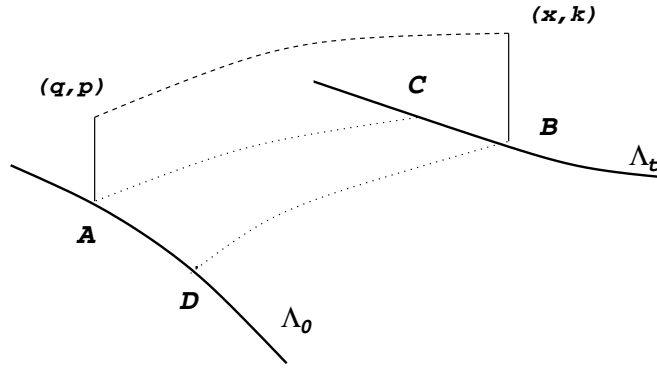


FIG. 3. Evolution of $k - S_x(x, t)$. In the figure are shown the points $A = (q, S'_0(q))$, $B = (x, S_x(x, t))$, $C = (\bar{x}, S_x(\bar{x}, t))$, and $D = (r, S'_0(r))$.

On the other hand,

$$x(\tau; q, p) - \bar{x}(\tau; q) = O(|x_p(\tau; q, S'_0(q))|(p - S'_0(q))) = O((p - S'_0(q))).$$

By the Hamiltonian system it follows that

$$\frac{dx_p}{dt} = k_p, \quad \frac{dk_p}{dt} = -V''(x)x_p, \quad x_p(0) = 0, \quad k_p(0) = 1,$$

and hence $x_p \approx t$ as $t \rightarrow 0$. Therefore, for small times and near the manifold, we have that $x(\tau; q, p) - \bar{x}(\tau; q) = O(\tau(p - S'_0(q)))$. Clearly, this relation is also valid for $0 \leq \tau \leq T$. Hence,

$$(3.13b) \quad x(\tau; q, p) - \bar{x}(\tau; q) = O(\tau(p - S'_0(q))).$$

Using (3.13a), (3.13b), and (3.12), we end up with

$$\exp\left(\int_0^t S_{xx}(x(\tau; q, p), \tau)\right) = J(\bar{x}, t)\left(1 + O\left(t^2(p - S'_0(q))\right)\right).$$

It then follows from (3.11) that

$$(3.14) \quad p - S'_0(q) = J(t; q) (k(t; q, p) - S_x(x(t; q, p), t)) + O\left(t^2(k - S_x(x, t))^2\right).$$

We next derive a formula for $S_{xxx}(\bar{x}(t, q), t) = S_{xxx}(x(t; q, S'_0(q)), t)$. Differentiation along the ray

$$\bar{\Gamma}_q = \{x = x(t; q, S'_0(q)), \quad k = k(t; q, S'_0(q))\}$$

yields

$$(3.15) \quad \frac{d}{dt} S_{xxx} = S_{xxx}t + S_x S_{xxxx},$$

since $k(t; q, S'_0(q)) = S_x(x(t; q, S'_0(q)), t)$. On the other hand, taking the ∂_x^3 -derivative of the eikonal equation we have

$$S_{xxx}t + 3S_{xx}S_{xxx} + S_x S_{xxxx} + V''' = 0.$$

From the last equation and (3.15) we see that

$$\frac{d}{dt}S_{xxx} = -3S_{xx}S_{xxx} - V''' ,$$

which, by taking into account (3.12), is written as

$$\frac{d}{dt}(J^3 S_{xxx}) = -J^3 V''' .$$

Integrating the last equation along the ray $\bar{\Gamma}_q$, we conclude that

$$(3.16) \quad S_0'''(q) = J^3(t; q)S_{xxx}(\bar{x}(t, q), t) + \int_0^t J^3(\tau; q)V'''(\bar{x}(\tau, q))d\tau,$$

with $J(t; q) = \frac{\partial \bar{x}}{\partial q}(t; q)$. Notice that since the Jacobian is calculated along the rays $\bar{x}(t; q) = x(t; q, S_0'(q))$, only the rays, and not the bicharacteristics, appear in (3.16). Since $\bar{x}(t; q)$ is close to $x(t; q, p)$ we have that

$$\begin{aligned} S_{xxx}(\bar{x}(t; q), t) &= S_{xxx}(x(t; q, p), t) + O(|x(t; q, p) - \bar{x}(t; q, p)|) \\ &= S_{xxx}(x(t; q, p), t) + O\left(t(k - S_x(x(t; q, p), t))\right). \end{aligned}$$

Hence, from (3.16) we obtain

$$(3.17) \quad \begin{aligned} S_0'''(q) &= J^3(t; q)S_{xxx}(x(t; q, p), t) + \int_0^t J^3(\tau; q)V'''(\bar{x}(\tau; q))d\tau \\ &\quad + O\left(t(k - S_x(x(t; q, p), t))\right). \end{aligned}$$

In a similar manner we can obtain equations for higher-order derivatives of S .

Remark. When working with the WKB solution, the Jacobian that enters naturally in the computations is not $J(\tau; q)$ but is $J(\tau; r)$ along the ray $\bar{x}(\tau; r)$, $0 \leq \tau \leq t$, for which $\bar{x}(t; r) = x$; see DB in Figure 3. Using the estimate

$$x(\tau; q, p) - \bar{x}(\tau; r) = O\left((\tau - t)(k - S_x(x, t))\right)$$

instead of (3.13b), we can rederive (3.14) and (3.17) with $J(\tau; r)$ in place of $J(\tau; q)$.

3.3. Solution of the limit Wigner equation. Using the evolution formulas (3.14) and (3.17), we now derive the solution W^0 of the limit Wigner equation

$$(3.18) \quad \frac{d}{dt}f^0(x, k, t) = f_t^0 + kf_x^0 - V'(x)f_k^0 = 0,$$

with initial data (cf. (2.11b))

$$(3.19) \quad f_0^0(q, p) = A_0^2(q) \delta(p - S_0'(q)).$$

Integrating (3.18)–(3.19) we obtain

$$(3.20) \quad f^0(x, k, t) = f_0^0(q(x, k, t), p(x, k, t)) = A_0^2(q(x, k, t)) \delta\left(p(x, k, t) - S_0'(q(x, k, t))\right).$$

Using (3.14) and then (1.11), we have

$$(3.21) \quad \begin{aligned} f^0(x, k, t) &= A_0^2(q(x, k, t))J^{-1}(x, t) \delta(k - S_x(x, t)) \\ &= A^2(x, t) \delta(k - S_x(x, t)). \end{aligned}$$

Note that (3.21) coincides with the limit Wigner function of the WKB solution (cf. (2.11a)). Also (3.20) is valid even in the multiphase case (see, however, section 4).

3.4. Approximate solution of the Wigner equation. Here we proceed to the construction of asymptotic approximations of the solutions of the (full) Wigner equation (3.1), which we denote by \tilde{f}^ϵ . We first consider an interesting special case.

The case of nonessential potentials. We say that a potential $V(x)$ is nonessential if $V(x) = ax^2 + bx + c$, with $a, b, c \in \mathbb{R}$. This includes, in particular, the case of the harmonic oscillator ($V(x) = x^2$). For nonessential potentials, as is easily seen from (2.9), the Wigner equation coincides with the limit Wigner equation, since $V^{(2m+1)}(x) \equiv 0, m = 1, 2, \dots$. Thus, the solution is given by

$$(3.22) \quad f^\epsilon(x, k, t) = f_0^\epsilon(q(x, k, t), p(x, k, t)),$$

where f_0^ϵ is the Wigner transform of the WKB initial datum (1.2).

The Wigner transform $W^\epsilon(x, k, t)$ of the WKB solution is approximated by $\widetilde{W}^\epsilon(x, k, t)$ (cf. (3.9)), while the Wigner transform of the initial data is approximated by

$$(3.23) \quad \widetilde{W}_0^\epsilon(q, p) = \frac{2}{\epsilon^{\frac{2}{3}}} \frac{A_0^2(q)}{|S_0'''(q)|^{\frac{1}{3}}} Ai\left(-\frac{2}{\epsilon^{\frac{2}{3}}} \frac{p - S_0'(q)}{(S_0'''(q))^{\frac{1}{3}}}\right).$$

We want to compare $\widetilde{W}_0^\epsilon(q(x, k, t), p(x, k, t))$, that is, the evolution of the Airy asymptotics of the initial data, with $\widetilde{W}^\epsilon(x, k, t)$. To this end we will use the evolution relations (3.14) and (3.17). We recall that according to (3.14) for p close to $S_0'(q)$,

$$p - S_0'(q) = (k - S_x(x, t))J(t; q) + O(t^2(k - S_x)^2).$$

On the other hand, for $V''' \equiv 0$, (3.17) takes the simple form

$$(3.24) \quad S_0'''(q) = J^3(t; q)S_{xxx}(x, t) + O(t(k - S_x(x, t))).$$

It then follows that for p close to $S_0'(q)$

$$\frac{p - S_0'(q)}{(S_0'''(q))^{\frac{1}{3}}} = \frac{k - S_x(x, t)}{S_{xxx}^{1/3}(x, t)} \left(1 + O(t(k - S_x(x, t)))\right),$$

whereas from (1.11) and (3.24) we similarly have that

$$\frac{A_0^2(q)}{(S_0'''(q))^{\frac{1}{3}}} = \frac{A^2(x, t)}{S_{xxx}^{1/3}(x, t)} \left(1 + O(t(k - S_x(x, t)))\right).$$

From the last two relations and (3.23), we have that

$$(3.25) \quad \begin{aligned} \widetilde{W}_0^\epsilon(q(x, k, t), p(x, k, t)) &= \frac{2}{\epsilon^{\frac{2}{3}}} \frac{A^2(x, t)}{S_{xxx}^{1/3}(x, t)} \left(1 + O(t(k - S_x(x, t)))\right) \\ &\quad \times Ai\left(-\frac{2}{\epsilon^{\frac{2}{3}}} \frac{k - S_x(x, t)}{S_{xxx}^{1/3}(x, t)} \left(1 + O(t(k - S_x(x, t)))\right)\right). \end{aligned}$$

It is remarkable that omitting the error terms in the right-hand side of (3.25) we recover $\widetilde{W}^\epsilon(x, k, t)$; cf. (3.9). Motivated by this we define

$$(3.26) \quad \tilde{f}^\epsilon(x, k, t) := \widetilde{W}_0^\epsilon(q(x, k, t), p(x, k, t)).$$

An easy analysis shows that $\tilde{f}^\epsilon(x, k, t) = (1 + o(1))\widetilde{W}^\epsilon(x, k, t)$ if $(k - S_x) = o(\epsilon^{\frac{2}{3}})$, whereas $\tilde{f}^\epsilon(x, k, t) = (1 + O(1))\widetilde{W}^\epsilon(x, k, t)$ if $(k - S_x) = O(\epsilon^{\frac{2}{3}})$. Hence, the Hamiltonian flow preserves the Airy asymptotics of the WKB solution locally near the Lagrangian manifold. In view of this, we may think of \tilde{f}^ϵ as defined in (3.26) as an approximation of the solution of the Wigner equation.

It is easy to check that as $\epsilon \rightarrow 0$

$$(3.27) \quad \tilde{f}^\epsilon(x, k, t) \rightarrow A^2(x, t)\delta(k - S_x(x, t)).$$

Moreover, \tilde{f}^ϵ satisfies approximately the projection formula (2.2). To see this we first note that for (x, t) fixed and ϵ small

$$(3.28) \quad \int_{-\infty}^{\infty} \tilde{f}^\epsilon(x, k, t) dk = \int_{-\infty}^{\infty} \widetilde{W}_0^\epsilon(q(x, k, t), p(x, k, t)) dk \approx \int_{S_x - \delta}^{S_x + \delta} \widetilde{W}_0^\epsilon(q(x, k, t), p(x, k, t)) dk$$

for some δ small (independent of ϵ), since the main contribution to the integral will come from the neighborhood of the point $k = S_x$. Also, along a vertical line $x = \text{const}$, $0 = dx = x_p dp + x_q dq$, whence $dq = -(x_p/x_q) dp$, and using the incompressibility condition in phase space

$$x_q k_p - x_p k_q = 1,$$

we obtain that $dk = (1/x_q) dp$. Since for k close to S_x there holds $x_q \approx J(t; r)$, we have

$$\begin{aligned} \int_{S_x - \delta}^{S_x + \delta} \widetilde{W}_0^\epsilon dk &= \int_{S'_0 - \delta_0}^{S'_0 + \delta_0} \frac{\widetilde{W}_0^\epsilon}{x_q} dp \approx \frac{1}{J(t; r)} \int_{S'_0 - \delta_0}^{S'_0 + \delta_0} \widetilde{W}_0^\epsilon dp \\ &\approx \frac{1}{J(t; r)} \int_{-\infty}^{\infty} \widetilde{W}_0^\epsilon dp = \frac{A_0^2(q)}{J(t; r)} = A^2(x, t) \end{aligned}$$

for some δ_0 small which is also independent of ϵ . In conclusion, as $\epsilon \rightarrow 0$

$$(3.29) \quad \int_{-\infty}^{\infty} \tilde{f}^\epsilon(x, k, t) dk = A^2(x, t) + o(1).$$

We point out that in the stream of calculations leading to (3.29), the fact that the potential is nonessential has not been used. By its definition, \tilde{f}^ϵ is simply transported as the exact solution, that is,

$$(3.30) \quad \tilde{f}^\epsilon(x, k, t) = \tilde{f}_0^\epsilon(q(x, k, t), p(x, k, t)), \quad \tilde{f}_0^\epsilon(q, p) = \widetilde{W}_0^\epsilon(q, p).$$

In addition to properties (3.28), (3.29), \tilde{f}^ϵ as given by (3.30) remains meaningful even at the caustics.

Remark. In our previous calculations we have assumed that we are away from degenerate points, that is, away from points $(q, S'_0(q))$ for which $S''_0(q) = 0$, and

away from points $(x, S_x(x, t))$ for which $S_{xxx}(x, t) = 0$, with $x = x(t; q, S'_0(q))$. For nonessential potentials, degenerate points $(q, S'_0(q))$ are moved by the Hamiltonian flow to degenerate points for all times; this can be easily seen from (3.17) (with $V'''(x) \equiv 0$). By replacing at the degenerate points the Airy function with the Dirac function, we may think of \tilde{f}^ϵ as being well defined everywhere. In this case, of course, at the degenerate points we lose the Airy asymptotics, and we simply recover the limit Wigner function.

The general case. The formula for \widetilde{W}^ϵ (cf. (3.9)) as well as for \widetilde{W}_0^ϵ (cf. (3.23)) both remain the same if an “essential potential” V is present in the Wigner equation. This does not mean, however, that (3.25) is valid anymore, since the simple relation (3.24) is not true in this case. If we use (3.17) instead of (3.24) we can write $\widetilde{W}_0^\epsilon(q, p)$ in terms of (x, k, t) in the form

$$(3.31) \quad \begin{aligned} \widetilde{W}_0^\epsilon(q(x, k, t), p(x, k, t)) &= \frac{2}{\epsilon^{2/3}} \frac{A^2(x, t) \left(1 + O\left(t(k - S_x)\right)\right)}{|S'''_{xxx} + J^{-3}(t; q) \int_0^t J^3 V''' d\tau|^{1/3}} \\ &\times Ai\left(-\frac{2}{\epsilon^{2/3}} \frac{k - S_x(x, t)}{(S'''_{xxx} + J^{-3}(t; q) \int_0^t J^3 V''' d\tau)^{1/3}} \left(1 + O\left(t(k - S_x)\right)\right)\right), \end{aligned}$$

which is the analogue of (3.25). However, (3.31) is not convenient for the construction of the approximate solution f^ϵ . For this reason we start from the similar formula

$$(3.32) \quad \begin{aligned} \widetilde{W}^\epsilon(x, k, t) &= \frac{2}{\epsilon^{2/3}} \frac{A_0^2(q)}{|S'''_0(q) - \int_0^t J^3 V''' d\tau|^{1/3}} \left(1 + O\left(t(p - S'_0(q))\right)\right) \\ &\times Ai\left(-\frac{2}{\epsilon^{2/3}} \frac{p - S'_0(q)}{(S'''_0(q) - \int_0^t J^3 V''' d\tau)^{1/3}} \left(1 + O\left(t(p - S'_0(q))\right)\right)\right), \end{aligned}$$

which is easily derived by solving (3.17) for S_{xxx} and using (3.14).

By omitting the error terms in the right-hand side of (3.32), we define \tilde{f}^ϵ as

$$(3.33) \quad \tilde{f}^\epsilon(x, k, t) := \frac{2}{\epsilon^{2/3}} \frac{A_0^2(q)}{|S'''_0(q) - \int_0^t J^3 V''' d\tau|^{1/3}} Ai\left(-\frac{2}{\epsilon^{2/3}} \frac{p - S'_0(q)}{(S'''_0(q) - \int_0^t J^3 V''' d\tau)^{1/3}}\right).$$

As in the case of nonessential potentials, an easy analysis shows that $\tilde{f}^\epsilon(x, k, t) = (1 + o(1))\widetilde{W}^\epsilon(x, k, t)$ if $(k - S_x) = o(\epsilon^{\frac{2}{3}})$. Moreover, \tilde{f}^ϵ has the correct limiting behavior; that is, (3.27) holds. Finally, \tilde{f}^ϵ thus defined satisfies (3.29), as it can be shown by the same argument as before, simply by replacing \widetilde{W}_0^ϵ in (3.28) by the right-hand side of (3.33).

We next want to find the evolution law of \tilde{f}^ϵ , that is, to find the analogue of (3.30). To this end we will use the following convolution identity:

$$(3.34) \quad \frac{1}{|b^3 - c^3|^{1/3}} Ai\left(\frac{x}{(b^3 - c^3)^{1/3}}\right) = \int_{-\infty}^{+\infty} \frac{1}{|b|} Ai\left(\frac{x - y}{b}\right) \frac{1}{|c|} Ai\left(-\frac{y}{c}\right) dy,$$

which is easily derived using the integral representation of the Airy function. By our usual convention, if some of the denominators in (3.34) are zero, we replace the corresponding Airy function with the Dirac function.

We will take

$$a = S'_0(q), \quad x = -\frac{2}{\epsilon^{2/3}}(p - a), \quad b^3 = S'''_0(q),$$

$$c^3 = \int_0^t J^3(\tau; q)V'''(x(\tau; q)) d\tau = \int_0^t J^3V''' d\tau.$$

Multiplying both sides of (3.34) by $\frac{2}{\epsilon^{2/3}}A_0^2(q)$ we have ($\xi = \epsilon^{2/3}y/2$)

$$\begin{aligned} & \frac{2}{\epsilon^{2/3}} \frac{A_0^2(q)}{|b^3 - c^3|^{1/3}} Ai\left(-\frac{2}{\epsilon^{2/3}} \frac{p - a}{(b^3 - c^3)^{1/3}}\right) \\ (3.35) \quad &= \left(\frac{2}{\epsilon^{2/3}}\right)^2 \frac{A_0^2(q)}{bc} \int_{-\infty}^{+\infty} Ai\left(-\frac{2}{\epsilon^{2/3}} \frac{(p - a) - \xi}{b}\right) Ai\left(-\frac{2}{\epsilon^{2/3}} \frac{\xi}{c}\right) d\xi \\ &= \frac{2A_0^2(q)}{\epsilon^{2/3} |b|} Ai\left(-\frac{2}{\epsilon^{2/3}} \frac{(p - a)}{b}\right) *_p \frac{2}{\epsilon^{2/3} |c|} Ai\left(-\frac{2}{\epsilon^{2/3}} \frac{p}{c}\right). \end{aligned}$$

Then, in view of (3.23) and (3.33), the identity (3.35) is written as

$$(3.36) \quad \tilde{f}^\epsilon(x, k, t) = \tilde{f}_0^\epsilon(q, p) *_p G^\epsilon(q, p, t),$$

with $\tilde{f}_0^\epsilon(q, p) = \widetilde{W}_0^\epsilon(q, p)$ and

$$(3.37) \quad G^\epsilon(q, p, t) := \frac{2}{\epsilon^{2/3}} \frac{1}{|\int_0^t J^3V''' d\tau|^{1/3}} Ai\left(-\frac{2}{\epsilon^{2/3}} \frac{p}{(\int_0^t J^3V''' d\tau)^{1/3}}\right).$$

Formula (3.36) shows that \tilde{f}_0^ϵ is not simply transported along the bicharacteristics, but it is also dispersed in the k -direction. Thus, \tilde{f}^ϵ is made up of two parts. First, the initial Airy function (\tilde{f}_0^ϵ) is transported by the Hamiltonian flow bringing Λ_0 to Λ_t . Second, the transported Airy function is “dispersed” as a result of convolution with a second Airy function (G^ϵ) that reflects the effect of the (“essential” part of the) potential. This process results in a new Airy function (\tilde{f}^ϵ) which is still concentrated on Λ_t , but with modified argument, and with asymptotics which are in agreement with the WKB solution.

In view of (3.36) we may think of $G^\epsilon(q, p, t)$ as the local asymptotic approximation of the fundamental solution of the Wigner equation (2.9). Notice that if $\epsilon \rightarrow 0$, then $G^\epsilon(q, p, t) \rightarrow \delta(p)$. Hence $\tilde{f}^0(q, p) = \tilde{f}_0^0(q, p) *_p \delta(p) = \tilde{f}_0^0(q, p)$, thus recovering (3.20). By the same reasoning, if $V''' \equiv 0$ we recover (3.30), whereas for $t = 0$ we find $\tilde{f}^\epsilon(x, k, t = 0) = \tilde{f}_0^\epsilon(q, p)$.

As usual, near degenerate points we lose the Airy asymptotics. In contrast with the case of nonessential potentials, in the general case we have no a priori information about the position of these points. Near such points, we guess that one should look for more refined asymptotics in the form of generalized Airy functions; cf. (2.22).

Finally, as in the case of the nonessential potentials, formula (3.36) is meaningful even at caustics—if these caustics do not correspond to degenerate points. We extend the definition of degenerate point by saying that a degenerate point is a point where the curve Λ_t has zero curvature.

All our analysis so far has been single-valued. In order to get some insight of what one should expect in the multiphase case we consider in the next section some concrete multiphase examples.

4. Multiphase optics: Some case studies. As we have seen in section 3.4, for $V(x) = ax^2 + bx + c$ the Wigner equation is a Liouville equation that can be easily solved by characteristics, that is,

$$(4.1) \quad f^\epsilon(x, k, t) = f_0^\epsilon(q(x, k, t), p(x, k, t)).$$

To simplify subsequent calculations we consider the case $V(x) \equiv 0$. Then the rays emanating from $\bar{x} = r$ at $t = 0$ are given by

$$(4.2) \quad \bar{x}(t; r) = x(t; r, S_0'(r)) = S_0'(r)t + r.$$

The Jacobian is

$$(4.3) \quad J(t; r) = \partial \bar{x} / \partial r = 1 + tS_0''(r),$$

where $r = r(\bar{x}, t)$ is found solving (4.2), and consequently the solution of the transport equation (1.5) is

$$(4.4) \quad A(\bar{x}(t; r), t) = \frac{A_0(r)}{\sqrt{1 + tS_0''(r)}}.$$

Moreover, the phase is given by

$$(4.5) \quad S(\bar{x}(t; r), t) = \frac{1}{2}(S_0'(r))^2 t + S_0(r).$$

Finally, the bicharacteristic emanating from (q, p) at $t = 0$ is given by $x = q + pt$, $k = p$, so that

$$(4.6) \quad q = x - kt, \quad p = k.$$

Then from (4.1) the solution of the Wigner equation is

$$(4.7) \quad f^\epsilon(x, k, t) = f_0^\epsilon(x - kt, k).$$

Also, the initial Lagrangian manifold $\Lambda_0 = \{(q, p) : p = S_0'(q)\}$ is moved by the Hamiltonian flow to the Lagrangian manifold (see section 3.2)

$$\Lambda_t = \{(x, k) : k = S_0'(x - kt)\}.$$

The fact that f^ϵ is given by the simple relation (4.7) allows us to treat some examples in a simple and intuitive way.

4.1. Study of fold. Consider the initial data

$$(4.8) \quad S_0(q) = -\frac{q^3}{3}, \quad A_0(q) \equiv 1.$$

Then the rays are given by $\bar{x}(t; r) = -r^2 t + r$, and the caustic is a global fold in (x, t) , given by the hyperbola $xt = 1/4$. In the illuminated zone $xt \leq 1$, from each point (x, t) pass two rays, which emanate from the points $r = r_\pm(x, t) = (1 \mp \sqrt{1 - 4xt})/2t$. The corresponding Jacobians are $J^\pm := J(t; r_\pm(x, t)) = \pm\sqrt{1 - 4xt}$.

The Wigner transform of the initial data is calculated explicitly in terms of the Airy function,

$$f_0^\epsilon(q, p) = \frac{1}{\pi\epsilon} \int_{-\infty}^{+\infty} \exp\left(i\left(\frac{2}{3\epsilon} \frac{\sigma^3}{3} + \frac{2}{\epsilon}(p + q^2)\sigma\right)\right) d\sigma = \frac{2^{\frac{2}{3}}}{\epsilon^{\frac{2}{3}}} Ai\left(\frac{2^{\frac{2}{3}}(p + q^2)}{\epsilon^{\frac{2}{3}}}\right).$$

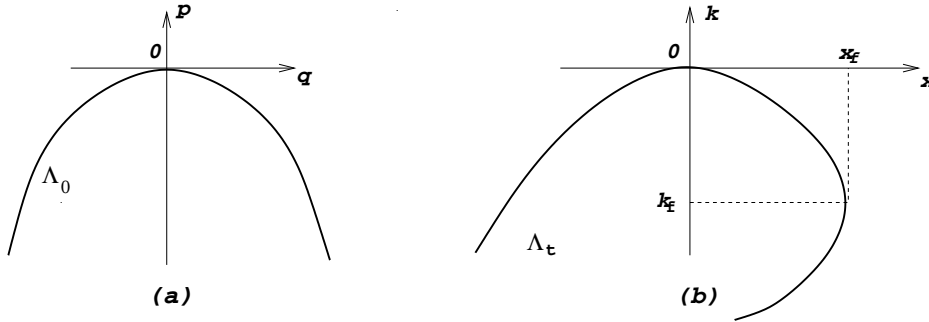


FIG. 4. Evolution of the Lagrangian manifold: (a) $t = 0$, initial position; (b) $t > 0$, formation of the fold ($x_f = 1/4t$, $k_f = 1/4t^2$).

By means of (4.7) the solution of the Wigner equation at any point (x, k, t) is given by

$$(4.9) \quad f^\epsilon(x, k, t) = \frac{2^{\frac{2}{3}}}{\epsilon^{\frac{2}{3}}} Ai\left(\frac{2^{\frac{2}{3}}(k^2 t^2 + (1 - 2xt)k + x^2)}{\epsilon^{\frac{2}{3}}}\right).$$

As $\epsilon \rightarrow 0$, we see that f^ϵ is an Airy function “concentrated” on the Lagrangian manifold

$$\Lambda_t = \{(x, k) : k^2 t^2 + (1 - 2xt)k + x^2 = 0\}.$$

However, for $t > 0$, in contrast with the single-phase optics, Λ_t is no longer the graph of a single-valued function $k = S_x(x, t)$; see Figure 4(b). Instead, it consists of two single (real-valued) branches $k = S_x^+(x, t)$ and $k = S_x^-(x, t)$, both of which exist only in the illuminated zone ($xt < 1/4$):

$$(4.10) \quad S_x^\pm(x, t) = \frac{-1 + 2xt \pm \sqrt{1 - 4xt}}{2t^2}.$$

In the shadow zone ($xt > 1/4$), S_x^\pm are complex-valued. Thus, (4.9) can be written as

$$(4.11) \quad f^\epsilon(x, k, t) = \frac{2^{\frac{2}{3}}}{\epsilon^{\frac{2}{3}}} Ai\left(\frac{2^{\frac{2}{3}}}{\epsilon^{\frac{2}{3}}} t^2 (k - S_x^-(x, t))(k - S_x^+(x, t))\right).$$

On the caustic $x_f = 1/4t$, we observe that $S_x^-(x_f, t) = S_x^+(x_f, t) = -1/4t^2 \equiv k_f$, and the expression defining Λ_t becomes a perfect square in k , which is the typical behavior of Lagrangian manifolds near folds. Clearly, at these points, $\frac{dx}{dk} = 0$, and the manifold Λ_t turns vertically.

The amplitude $|\psi^\epsilon(x, t)|$ is given by (cf. (2.2))

$$(4.12) \quad |\psi^\epsilon(x, t)|^2 = \int_{-\infty}^{+\infty} f^\epsilon(x, k, t) dk.$$

This can be explicitly computed by means of the following “projection identity” (Appendix B, (B.1)):

$$\int_{-\infty}^{+\infty} Ai(ak^2 + bk + c) dk = \frac{2\pi}{2^{\frac{1}{3}} \sqrt{a}} Ai^2\left(-\frac{b^2 - 4ac}{4^{5/3} a}\right), \quad a > 0.$$

Then for f^ϵ as in (4.9) we have that $a = (\frac{2}{\epsilon})^{2/3}t^2$, $b = (\frac{2}{\epsilon})^{2/3}(1 - 2xt)$, $c = (\frac{2}{\epsilon})^{2/3}x^2$, and the amplitude at any point (x, t) is given by

$$(4.13) \quad |\psi^\epsilon(x, t)|^2 = \frac{2\pi}{\epsilon^{1/3}t} Ai^2\left(-\frac{1}{\epsilon^{2/3}t^2} \frac{(1 - 4xt)}{4}\right).$$

We note that in the illuminated zone $xt < 1/4$, the argument of $Ai^2(\cdot)$ in (4.13) is negative, and therefore $|\psi^\epsilon|$ is oscillatory there, whereas in the shadow zone $xt > 1/4$, and $|\psi^\epsilon|$ is exponentially decreasing as $\epsilon \rightarrow 0$. Finally, on the caustic $xt = 1/4$,

$$\left|\psi^\epsilon\left(\frac{1}{4t}, t\right)\right| = \frac{\sqrt{2\pi}Ai(0)}{\sqrt{t}\epsilon^{1/6}} = \frac{\Gamma(1/3)}{3^{1/6}\sqrt{2\pi}\sqrt{t}\epsilon^{1/6}} = O(\epsilon^{-1/6}).$$

This is the typical picture of a high-frequency wave field ($\epsilon \rightarrow 0$) in the region of a fold [GS, Chap. VII, sect. 6].

The limit Wigner function. We now turn our attention to the limit Wigner function. We use once more the fact that $(1/\epsilon)Ai(y/\epsilon) \rightarrow \delta(y)$ as $\epsilon \rightarrow 0$ to take the limit in (4.11). It then follows that

$$(4.14a) \quad f^\epsilon(x, k, t) \rightarrow f^0(x, k, t) = \frac{1}{t^2} \delta\left((k - S_x^-)(k - S_x^+)\right).$$

In the illuminated zone, $S_x^- \neq S_x^+$ and both are real-valued; hence

$$f^0(x, k, t) = \frac{1}{t^2} \frac{\delta(k - S_x^+) + \delta(k - S_x^-)}{|S_x^+ - S_x^-|}, \quad xt < \frac{1}{4}.$$

Using (4.10) we see that

$$t^2|S_x^+ - S_x^-| = \sqrt{1 - 4xt} = |J^\pm|.$$

Also,

$$\frac{1}{|J^\pm|} = \frac{A_0^2(r_\pm)}{|J^\pm|} =: A_\pm^2(x, t).$$

Thus, in the illuminated zone, we have that

$$(4.14b) \quad f^0(x, k, t) = A_+^2(x, t)\delta(k - S_x^+) + A_-^2(x, t)\delta(k - S_x^-), \quad xt < 1/4.$$

In the shadow zone, since the argument of the Airy function in (4.11) has no real roots, we obtain

$$(4.14c) \quad f^0(x, k, t) = 0, \quad xt > 1/4.$$

By means of (4.12) and with f^0 in place of f^ϵ , we can compute the limit amplitude

$$(4.15) \quad |\psi^0(x, t)|^2 = \begin{cases} A_+^2(x, t) + A_-^2(x, t), & xt < 1/4, \\ 0, & xt > 1/4, \end{cases}$$

thus recovering the two-phase geometrical optics; see Appendix D.

Finally, let us see which is the limiting behavior of f^ϵ on the fold. As we mentioned above, there holds $S_x^+(x_f, t) = S_x^-(x_f, t) =: k_f$. Hence, from (4.14a) we formally write

$$(4.16) \quad f^0(x, k, t) = \frac{1}{t^2} \delta\left((k - k_f)^2\right), \quad xt = 1/4, \quad x = x_f.$$

The Dirac function in the right-hand side of (4.16) is not well defined. One can see this by noting that if $\phi_\epsilon(x)$ is a sequence that tends to the Dirac mass (as ϵ tends to zero), then the limit of $\phi_\epsilon(x^2)$ is not uniquely defined but depends on the sequence itself. This explains why the limit Wigner function f^0 is not well defined on the fold.

We then proceed directly from (4.11) (with $S_x^- = S_x^+ = k_f$):

$$(4.17) \quad f^\epsilon(x, k, t) = \frac{2^{\frac{2}{3}}}{\epsilon^{\frac{2}{3}}} Ai\left(\frac{2^{\frac{2}{3}}}{\epsilon^{\frac{2}{3}}} t^2 (k - k_f)^2\right), \quad x = x_f = \frac{1}{4t}.$$

Using the weak limit (see Appendix B, (B.4))

$$(1/\sqrt{\eta}) Ai(y^2/\eta) \rightarrow \frac{\Gamma^2(1/3)}{2^{1/3} 3^{1/3} 2\pi} \delta(y), \quad \eta \rightarrow 0,$$

we obtain that

$$\epsilon^{1/3} f^\epsilon(x, k, t) \rightarrow \frac{\Gamma^2(1/3)}{2\pi t 3^{1/3}} \delta(k - k_f), \quad x = x_f, \quad \epsilon \rightarrow 0,$$

where $\Gamma(\cdot)$ denotes the gamma function.

The local asymptotics of f^ϵ . Here we compare the exact solution f^ϵ with \widetilde{W}^ϵ and \widetilde{f}^ϵ that we defined in section 3 (cf. (3.9), (3.22), (3.26)). The formula (4.11) for f^ϵ is valid for all (x, k, t) without any smallness assumption on ϵ . For ϵ small and x away from the caustic, we can approximate f^ϵ , say, near the upper branch $k \approx S_x^+$, as follows:

$$f^\epsilon(x, k, t) \approx \frac{2^{\frac{2}{3}}}{\epsilon^{\frac{2}{3}}} Ai\left(\frac{2^{\frac{2}{3}}}{\epsilon^{\frac{2}{3}}} t^2 (S_x^+ - S_x^-)(k - S_x^+)\right).$$

Using (4.10) we have

$$S_x^+ - S_x^- = \frac{J^+}{t^2}, \quad S_{xxx}^+ = -\frac{2}{(J^+)^3}, \quad (A^+(x, t))^2 = \frac{A_0^2(r_+)}{|J^+|} = \frac{|S_{xxx}^+|^{1/3}}{2^{1/3}}.$$

In view of these relations, near the upper branch $k = S_x^+$ of the manifold Λ_t , we have

$$(4.18) \quad f^\epsilon(x, k, t) \approx \frac{2^{\frac{2}{3}}}{\epsilon^{\frac{2}{3}}} Ai\left(\frac{2^{\frac{2}{3}}}{\epsilon^{\frac{2}{3}}} J^+(k - S_x^+)\right) = \frac{2}{\epsilon^{\frac{2}{3}}} \frac{(A^+(x, t))^2}{|S_{xxx}^+|^{\frac{1}{3}}} Ai\left(\frac{2}{\epsilon^{\frac{2}{3}}} \frac{k - S_x^+}{(S_{xxx}^+)^{\frac{1}{3}}}\right).$$

We note that the right-hand side of this is the Airy asymptotics of the WKB solution, which we denoted by $\widetilde{W}^\epsilon(x, k, t)$; cf. (3.9).

On the other hand, if we recall the definition of \widetilde{f}^ϵ (cf. (3.26)) it is easy to see that $\widetilde{f}^\epsilon(x, k, t) = f^\epsilon(x, k, t)$; that is, in this simple example \widetilde{f}^ϵ coincides with the exact solution.

We finally make some interesting remarks concerning the phase-space formulas derived in section 3.2. Let $x \leq 1/4t$ be a point in the physical space. From this point pass two rays which emanate from the points r_\pm . We consider the points (x, k) in phase space, and let $(q(x, k, t), p(x, k, t))$ be given by (4.6). In particular, $q(x, S_x^\pm, t) = r_\pm$. An easy calculation shows that

$$(4.19) \quad p - S_0'(q) = p + q^2 = J^+(k - S_x^+) + t^2(k - S_x^+)^2.$$

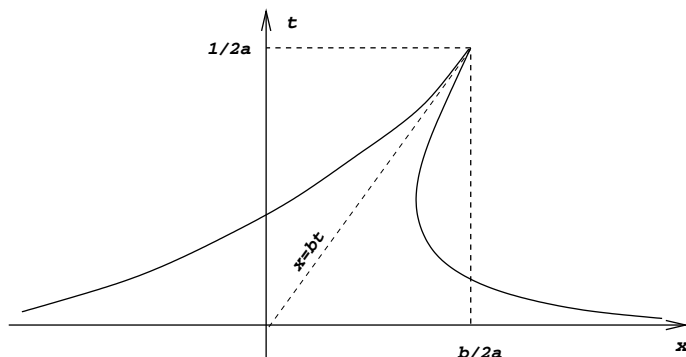


FIG. 5. Geometry of the cusp: $27(\frac{x}{t} - b)^2 = 4(\frac{1}{t} - 2a)^3$.

Formula (4.19) remains true if we replace J^+ and S_x^+ by J^- and S_x^- , respectively. This is to be compared with (3.14) along with the remark at the end of section 3.2. From (4.19) it follows that

$$\tilde{f}_\epsilon(x, k, t) = \frac{2^{\frac{2}{3}}}{\epsilon^{\frac{2}{3}}} Ai\left(\frac{2^{\frac{2}{3}}}{\epsilon^{\frac{2}{3}}}(J^\pm(k - S_x^\pm) + t^2(k - S_x^\pm)^2)\right).$$

Away from the caustic, and for k close to S_x^+ , $J^+ \neq 0$, and the first term $J^+(k - S_x^+)$ is dominant. Thus, near the upper branch, we recover (4.18). On the other hand, at the caustic we have that $J^\pm = 0$ and $S_x^+ = S_x^- = k_f$, thus recovering (4.17). We observe that the WKB solution corresponds to the linear term in the argument of the Airy function. Therefore, geometrical optics follows by a suitable linearization in phase space near Λ_t .

4.2. Study of cusp. We now consider the initial data

$$(4.20) \quad S_0(q) = -\frac{q^4}{4} - aq^2 + bq, \quad a > 0, \quad b > 0, \quad A_0(q) \equiv 1.$$

Then the rays are given by $\bar{x}(t; r) = -r^3t + (1 - 2at)r + bt$, and the caustic is the cusp $27u^2 = 4v^3$, where $u = x/t - b$ and $v = 1/t - 2a$. The beak $u = v = 0$ of the cusp is the point $(x_b = b/2a, t_b = 1/2a)$; see Figure 5.

The Wigner transform of the initial data is explicitly given by

$$(4.21) \quad f_0^\epsilon(q, p) = \frac{1}{\pi\epsilon} \int_{-\infty}^{+\infty} \exp\left(\frac{2i}{\epsilon}(q\sigma^3 + (q^3 + 2aq + p - b)\sigma)\right) d\sigma.$$

For $q \neq 0$, it is calculated again in terms of the Airy function,

$$(4.22a) \quad f_0^\epsilon(q, p) = \frac{2^{\frac{2}{3}}}{\epsilon^{\frac{2}{3}} |3q|^{\frac{1}{3}}} Ai\left(\frac{2^{\frac{2}{3}}(q^3 + 2aq + p - b)}{\epsilon^{\frac{2}{3}}(3q)^{\frac{1}{3}}}\right), \quad q \neq 0,$$

while for $q = 0$ it is a Dirac mass,

$$(4.22b) \quad f_0^\epsilon(q, p) = \delta(p - b), \quad q = 0.$$

Although $q = 0$ is a singular point of the initial Wigner function $W_0^\epsilon(q, p)$, using the formula $1/\epsilon Ai(y/\epsilon) \rightarrow \delta(y)$ as $\epsilon \rightarrow 0$ we see that

$$f_0^\epsilon(q, p) \rightarrow f_0^\epsilon(0, p), \quad q \rightarrow 0.$$

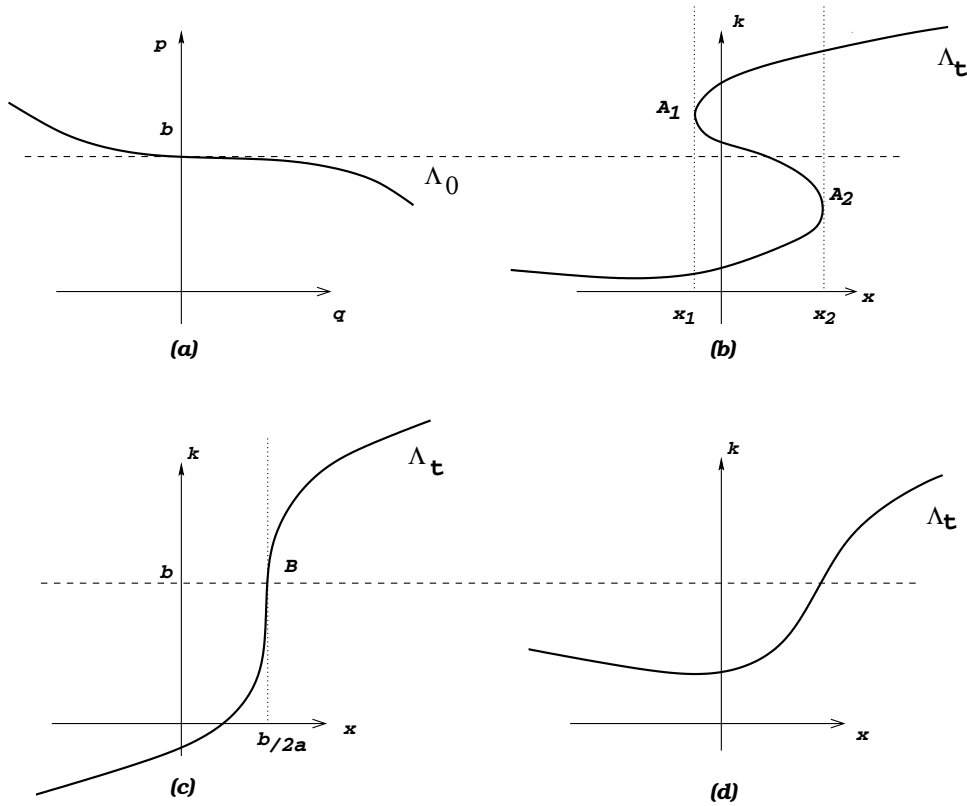


FIG. 6. Evolution of the Lagrangian manifold: (a) $t = 0$, initial position; (b) $0 < t < 1/2a$, formation of cusp: points $A_1(x_1, k_f(x_1))$ and $A_2(x_1, k_f(x_1))$ project onto the fold sides of the cusp; (c) $t = 1/2a$, formation of the beak of the cusp; (d) $t > 1/2a$, single-phase evolution.

By means of (4.7) the Wigner function at any point (x, k, t) is given by

(4.23a)

$$f^\epsilon(x, k, t) = \frac{1}{\pi\epsilon} \int_{-\infty}^{+\infty} \exp\left(\frac{2i}{\epsilon}((x - kt)\sigma^3 + (x - kt)^3 + 2a(x - kt) + k - b)\sigma\right) d\sigma$$

or, alternatively, by

$$(4.23b) \quad f^\epsilon(x, k, t) = \begin{cases} \frac{2^{\frac{2}{3}}}{\epsilon^{\frac{2}{3}} |3(x-kt)|^{\frac{1}{3}}} Ai\left(\frac{2^{\frac{2}{3}}((x-kt)^3 + 2a(x-kt) + k - b)}{\epsilon^{\frac{2}{3}} (3(x-kt))^{\frac{1}{3}}}\right), & x \neq kt, \\ \delta(k - b), & x = kt. \end{cases}$$

As $\epsilon \rightarrow 0$, f^ϵ is again concentrated on the Lagrangian manifold

$$\Lambda_t = \{(x, k) : k^3 t^3 - 3xt^2 k^2 - (1 - 3tx^2 - 2at)k - x^3 - 2ax + b = 0\}.$$

The evolution of Λ_t is shown in Figure 6. We note that for $0 < t < 1/2a$, there is a region $(x_1(t) < x < x_2(t))$ where this manifold consists of three branches. Notice that points $x_1(t)$ and $x_2(t)$ trace the fold sides of the cusp. At $t = 1/2a$, this region degenerates to the point $B = (x = b/2a, k = b)$, which is an inflection point of Λ_t

with vertical tangent. Point B projects onto the beak of the cusp. For $t > 1/2a$ the manifold becomes and stays thereafter single-valued.

The amplitude $\psi^\epsilon(x, t)$ at any point is given by the formula (4.12),

$$|\psi^\epsilon(x, t)|^2 = \int_{-\infty}^{+\infty} f^\epsilon(x, k, t) dk.$$

This can be again explicitly computed by means of a new “projection identity” (Appendix B, (B.5)):

$$\int_{-\infty}^{+\infty} \frac{1}{|\xi|^{1/3}} Ai\left(\frac{\lambda}{\xi^{1/3}}(\xi^3 - v\xi + u)\right) d\xi = \frac{1}{2\pi} \frac{1}{\sqrt{2}\lambda^{1/4}} |P(-V, U)|^2, \quad \lambda > 0,$$

where $P(-U, V)$ denotes the Pearcey integral [WO], [KAM]

$$P(-V, U) = \int_{\mathbb{R}} \exp\left(i\left(\frac{t^4}{4} - V\frac{t^2}{2} + Ut\right)\right) dt,$$

with

$$V = \frac{1}{\sqrt{2}}\lambda^{3/4}v, \quad U = \frac{1}{2^{3/4}}\lambda^{9/8}u.$$

Thus, we obtain

$$(4.24) \quad |\psi^\epsilon(x, t)|^2 = 3^{-3/4} \frac{2\pi}{t} \left(\frac{2}{\epsilon}\right)^{1/2} |P(-r, s)|^2,$$

with r, s defined by

$$(4.25) \quad r = \frac{1}{3^{1/4}\epsilon^{1/2}}v, \quad s = \frac{1}{3^{3/8}\epsilon^{3/4}}u.$$

We point out that formula (4.24) is valid at any point (x, t) as well as for any $\epsilon > 0$.

Using the asymptotic expansions of the Pearcey integral [KAM], we find that $|\psi^\epsilon(x, t)| = O(\epsilon^{-1/4})$ near the beak, and $|\psi^\epsilon(x, t)| = O(\epsilon^{-1/6})$ near the fold sides of the cusp, which is the typical behavior of a wave field near a cusp.

The limit Wigner function. There are three cases to be considered depending on the roots k_i ($i = 1, 2, 3$) of the equation

$$(4.26) \quad k^3 - \frac{3x}{t}k^2 - \frac{1}{t^3}(1 - 3tx^2 - 2at)k + \frac{1}{t^3}(-x^3 - 2ax + b) = 0.$$

Recall that these roots are given by the formulas

$$k_1 = P + T + \frac{x}{t}, \quad k_{2,3} = -\frac{1}{2}(P + T) + \frac{x}{t} \pm \frac{i\sqrt{3}}{2}(P - T),$$

where

$$P = (R + \sqrt{D})^{1/3}, \quad T = (R - \sqrt{D})^{1/3},$$

$$D = Q^3 + R^2, \quad Q = -v/3t^2, \quad R = u/2t^3.$$

Note that the discriminant D vanishes on the caustic $27u^2 = 4v^3$. Equation (4.26) has three distinguished roots for (x, t) inside the cusp; see Figure 5. On the fold sides of the cusp $u \neq 0$, it has a double root $k_2 = k_3 = k_f$. In this case

$$(4.27) \quad k_1(x, t) = 2 \frac{u^{1/3}}{2^{1/3}t} + \frac{x}{t}, \quad k_f(x, t) = -\frac{u^{1/3}}{2^{1/3}t} + \frac{x}{t}.$$

Finally, on the beak $u = v = 0$, (4.26) has a triple root $k_1 = k_2 = k_3 = k_b = b$.

Therefore, in the three-phase region inside the cusp, the limit Wigner function is given by

$$(4.28) \quad f^0(x, k, t) = \delta(t^3(k - k_1)(k - k_2)(k - k_3)) = \frac{1}{t^3} \sum_{\substack{i=1,2,3 \\ l \neq i, m \neq i}} \frac{\delta(k - k_i)}{|k_i - k_l| |k_i - k_m|},$$

whereas in the single-phase region outside the cusp, the limit Wigner function consists of a single Dirac function.

Similar results concerning the recovery of geometric optics, as for the case of the fold, can be derived for the limit Wigner function. Also, as in the case of the fold, on the caustic the limit Wigner function becomes singular, and the limiting behavior is as follows:

(1) On the fold sides of the cusp, $27u^2 = 4v^3 \neq 0$, $k_2 = k_3 = k_f$, and f^ϵ is given by

$$(4.29) \quad f^\epsilon(x, k, t) = \frac{2^{2/3}}{\epsilon^{2/3} 3^{1/3}} \frac{1}{(x - kt)^{1/3}} Ai \left(-\frac{2^{2/3}}{\epsilon^{2/3} 3^{1/3}} \frac{t^3(k - k_1)(k - k_f)^2}{(x - kt)^{1/3}} \right).$$

Then, for $k \approx k_f$, we approximate f^ϵ by

$$f^\epsilon(x, k, t) \approx \frac{2^{2/3}}{\epsilon^{2/3} 3^{1/3}} \frac{1}{(x - k_f t)^{1/3}} Ai \left(-\frac{2^{2/3}}{\epsilon^{2/3} 3^{1/3}} \frac{t^3(k_f - k_1)(k - k_f)^2}{(x - k_f t)^{1/3}} \right).$$

Using (4.27) and then (B.4), we find that

$$\epsilon^{1/3} f^\epsilon(x, k, t) \rightarrow \frac{3^{1/6} \Gamma^2(1/3)}{2\pi t v^{1/3}} \delta(k - k_f), \quad \epsilon \rightarrow 0.$$

(2) Finally, on the beak $u = v = 0$, and $k_i = k_b = b$, $i = 1, 2, 3$. It follows from (4.23a) that f^ϵ is given by

$$(4.30) \quad f^\epsilon(x, k, t) = \frac{1}{3^{1/3} t^{1/3}} \frac{2^{2/3}}{\epsilon^{2/3}} Ai \left(\frac{2^{2/3}}{\epsilon^{2/3}} \frac{1}{3^{1/3}} (k - k_b)^{8/3} \right) \frac{1}{|k - k_b|^{1/3}}.$$

Using formula (B.8) (cf. Appendix B),

$$\frac{1}{\eta^{1/4}} \frac{1}{|y|^{1/3}} Ai \left(\frac{y^{8/3}}{\eta} \right) \rightarrow \frac{3^{1/4} 2^{1/2} \Gamma^2(1/4)}{8\pi} \delta(y), \quad \eta \rightarrow 0,$$

we find

$$\epsilon^{1/2} f^\epsilon(x, k, t) \rightarrow \frac{\Gamma^2(1/4)}{4\pi t^{1/3}} \delta(k - k_b), \quad \epsilon \rightarrow 0.$$

The local asymptotics of f^ϵ . We first show that away from the caustic f^ϵ is approximated from \widetilde{W}^ϵ . To see this we consider a point (x, k) in the single-phase region and close to the branch $k \approx k_1 = S_{1,x}$. Let $r_1 = r_1(x, t)$ be the (unique) real root of the equation

$$(4.31) \quad r^3 t - (1 - 2at)r + x - bt = 0,$$

and denote by $J_1 = \frac{\partial \bar{x}}{\partial r_1}$ the Jacobian of the ray $\bar{x}(r_1, t)$. A straightforward calculation shows that

$$J(t; r_1) = -t^3(k_1 - k_2)(k_1 - k_3)$$

and

$$S_{1,xxx}(x, t) = k_{1,xx} = \frac{6(x - k_1 t)}{t^9(k_1 - k_2)^3(k_1 - k_3)^3}.$$

Then, for ϵ small, we approximate (4.23b) for $x - kt \neq 0$ as follows:

$$(4.32) \quad \begin{aligned} f^\epsilon(x, k, t) &\approx \frac{2^{\frac{2}{3}}}{\epsilon^{\frac{2}{3}} |3(x-kt)|^{\frac{1}{3}}} Ai\left(\frac{2^{\frac{2}{3}} J_1(k-k_1)}{\epsilon^{\frac{2}{3}} (3(x-kt))^{\frac{1}{3}}}\right) \\ &= \frac{2}{\epsilon^{\frac{2}{3}}} \frac{A_1^2(x,t)}{|S_{1,xxx}|^{1/3}} Ai\left(\frac{2}{\epsilon^{\frac{2}{3}}} \frac{k-S_{1,x}}{(S_{1,xxx})^{1/3}}\right), \end{aligned}$$

with $A_1^2(x, t) = 1/J_1 = A_0^2(r_1)/J_1$. We note that the second line of (4.32) is \widetilde{W}^ϵ .

Concerning \tilde{f}_ϵ it is easy to see that away from the points $x - kt = 0$, \tilde{f}_ϵ coincides with f^ϵ . The condition $(x - kt) \neq 0$ is natural since it excludes the degenerate points. Notice that the inflection point of the initial manifold ($q = 0, p = b$) propagates to the inflection point $(x = bt, k = b)$ for $t \geq 0$.

Let (x, k) be a point in the single-phase region, and let $k - S_{1,x} = k - k_1$ be its distance from Λ_t , with k_1 being the (unique) real root of (4.26). A straightforward calculation shows that

$$(4.33) \quad p - S'_0(q) = p + q^3 + 2aq - b = J_1(k - k_1) + 3t^2(x - k_1 t)(k - k_1)^2 - t^3(k - k_1)^3.$$

Formula (4.33) remains true even in the three-phase region. Moreover, in the three phase region, (4.33) is still correct if we replace (J_1, k_1) by either (J_2, k_2) or (J_3, k_3) , where $k_i, i = 1, 2, 3$, are the roots of (4.26), and $J_i = -3r_i^2 t + 1 - 2at$, with $r_i = r_i(x, t), i = 1, 2, 3$, are the roots of the cubic equation (4.31). Hence, we may write

$$(4.34) \quad \begin{aligned} \tilde{f}_\epsilon(x, k, t) &= \frac{2^{\frac{2}{3}}}{\epsilon^{\frac{2}{3}} |3q|^{\frac{1}{3}}} Ai\left(\frac{2^{\frac{2}{3}}(p - S'_0(q))}{\epsilon^{\frac{2}{3}}(3q)^{\frac{1}{3}}}\right) \\ &= \frac{2^{\frac{2}{3}}}{\epsilon^{\frac{2}{3}} |3(x - kt)|^{\frac{1}{3}}} Ai\left(\frac{2^{\frac{2}{3}}\left(J_i(k - k_i) + 3t^2(x - k_i t)(k - k_i)^2 - t^3(k - k_i)^3\right)}{\epsilon^{\frac{2}{3}}(3(x - kt))^{\frac{1}{3}}}\right). \end{aligned}$$

Away from the caustic and for k close to $k_i = S_{i,x}$ we have that $J_i \neq 0$ and the dominant term in the numerator of the argument of the Airy function in (4.35) is

the linear one, thus recovering (4.32). On the fold sides of the cusp, we have that $k_2 = k_3 = k_f$, $J_2 = J_3 = 0$, whereas $x - k_f t \neq 0$. Hence the dominant term is the quadratic one. Using (4.27) we easily see that $3t^2(x - k_f t) = t^3(k_f - k_1)$, and thus we recover (4.29). Finally, on the beak of the cusp only the cubic term survives, and we recover (4.30) by a suitable limiting argument. Again, the WKB approximation follows by “linearizing” (4.33).

4.3. Other cases. We close this section by making some comments concerning initial phases $S_0(q)$, which result in either degenerate or “higher-order” caustics.

A quadratic initial phase, e.g., $S_0(q) = -q^2/2$, with initial amplitude $A_0(q) = 1$, yields a focal point ($x = 0, t = 1$) from which pass all rays emanating from the x -axis at $t = 0$ (cf. [SMM, Ex. 1.2]). This degenerate case can be studied in detail. We note, in particular, that the solution of the Wigner equation is given by

$$(4.35) \quad f^\epsilon(x, k, t) = \frac{1}{|t-1|} \delta\left(k - \frac{x}{t-1}\right),$$

which in turn yields the singular wave amplitude $\psi^\epsilon(x, t) = \frac{1}{|t-1|}$ for any $x \in \mathbb{R}$.

On the other hand, one may choose as $S_0(q)$ a general polynomial in q of degree higher than four. In this case a detailed study of the exact solution is not possible, in general, due to obvious algebraic difficulties. Nevertheless, some interesting observations can still be made.

(i) The exact Wigner function (f^ϵ) is now given in the form of a generalized Airy function, that is,

$$(4.36) \quad f^\epsilon(x, k, t) = \frac{A^2(x, t)}{2\pi} \int_{-\infty}^{+\infty} \exp\left(i \sum_{m=0}^N a_m(x, k, t) \epsilon^{2m} \sigma^{2m+1}\right) d\sigma$$

for suitable coefficients $A(x, t)$, $a_m(x, k, t)$.

(ii) As in the cases of the fold or the cusp, caustics will again result from points of the Lagrangian manifold Λ_t having vertical tangents. There will be, however, some differences. Notice that near the fold, the manifold Λ_t is locally a parabola $x(k) \sim k^2$, whereas near the cusp Λ_t is locally cubic, that is, $x(k) \sim k^3$. In the general case, near the caustic the manifold will behave as $x(k) \sim k^n$. It is not difficult to see that for $n \geq 3$, at these points Λ_t has zero curvature; that is, they are degenerate points. In particular, the Airy asymptotics (\tilde{f}_ϵ) are not valid there.

(iii) Near these “higher-order” caustics, new projection identities hold (cf. [BEW]). These identities are similar to those in Appendix B, but they involve generalized Airy functions of the form (4.36). By means of these identities the wave amplitude can be computed explicitly in terms of suitable phase integrals.

Thus, although the Airy asymptotics of the Wigner function can capture the local scales near a fold or a cusp, more refined asymptotics are needed when approaching “higher-order” caustics.

5. Summary and concluding remarks. We summarize the picture that emerges from our previous analysis.

Given the Wigner transform W^ϵ of a WKB function $\psi^\epsilon(x) = A(x) \exp(iS(x)/\epsilon)$, we have defined the semiclassical Wigner function \widetilde{W}^ϵ as the Airy approximation of W^ϵ ; cf. (2.25). This approximation is valid away from degenerate points of the Lagrangian manifold $\Lambda = \{k = S'(x)\}$, that is, away from points of Λ of zero curvature.

We have seen that the Wigner transform of the WKB solution of (1.1), (1.2) is an approximate solution of the (full) Wigner equation (3.1), and therefore at each time $t > 0$, in the single-phase case, $\widetilde{W}^\epsilon(x, k, t)$ is an asymptotic approximation of the solution of the (full) Wigner equation.

In the special case of nonessential potentials ($V'''(x) \equiv 0$), we have seen that the Hamiltonian flow preserves the Airy asymptotics \widetilde{W}^ϵ , and we defined the asymptotic approximation (\tilde{f}^ϵ) of the solution of the Wigner equation as the evolution of \widetilde{W}_0^ϵ under the Hamiltonian flow:

$$(5.1) \quad \tilde{f}^\epsilon(x, k, t) = \widetilde{W}_0^\epsilon(q(x, k, t), p(x, k, t)), \quad (V'''(x) = 0).$$

In the general case a similar analysis lead to the following definition of \tilde{f}^ϵ :

$$(5.2) \quad \tilde{f}^\epsilon(x, k, t) = \widetilde{W}_0^\epsilon(q(x, k, t), p(x, k, t)) *_p G^\epsilon(q(x, k, t), p(x, k, t), t),$$

with G^ϵ given by (3.37). Formula (5.2) reduces to (5.1) when $V'''(x) \equiv 0$ and shows in a clear way the transport-dispersive character of the Wigner equation. By its construction \tilde{f}^ϵ is in agreement with the Airy asymptotics of the WKB solution; in addition, the above formulas are meaningful even near caustics (if these caustics do not correspond to degenerate points of Λ_t) where the WKB solution fails. The simple examples of section 4 support the validity of (5.1).

In the classical limit ($\epsilon = 0$), \tilde{f}^ϵ converges to the limit Wigner function f^0 . Initially, $f_0^0(q, p) = A_0^2(q)\delta(p - S_0'(q))$ is a Dirac mass concentrated on the Lagrangian manifold Λ_0 . Moreover, f^0 satisfies a simple transport equation, the limit Wigner equation (cf. (2.10)), which can be thought of as the zero dispersion limit of the (full) Wigner equation. The solution of the limit Wigner equation is given by

$$(5.3) \quad f^0(x, k, t) = A_0^2(q(x, k, t))\delta(p(x, k, t) - S_0'(q(x, k, t))).$$

Although formula (5.3) is formally valid everywhere, it is not well defined on the caustics, because, as the examples of section 4 show, the argument of the Dirac function ceases to have simple roots with respect to k . Thus, with the exception of caustic points, f^0 remains a Dirac mass concentrated on the Lagrangian manifold Λ_t , and this recovers the multiphase geometrical optics.

In the examples of section 4, it was found that on the fold, $\epsilon^{1/3}f^\epsilon$ tends to a Dirac mass, whereas on the beak of the cusp, $\epsilon^{1/2}f^\epsilon$ tends to a Dirac mass as ϵ tends to zero. This singular limiting behavior is, in a sense, expected, since the limit Wigner function has been designed to capture the oscillations at scale of order $O(\epsilon)$, whereas on caustics different scales appear. On the other hand, the semiclassical Wigner function conveys the necessary information and predicts the correct wave amplitude, even on caustics. This was accomplished by means of the projection formula

$$(5.4) \quad |\psi_\epsilon(x, t)|^2 = \int_{-\infty}^{+\infty} f^\epsilon(x, k, t) dk$$

in conjunction with suitable projection identities involving the Airy function (cf. Appendix B).

Thus, although the limit Wigner function is a simpler object to study, it conveys no information near caustics, and this motivates the search for asymptotic approximations of the Wigner function, such as \tilde{f}^ϵ defined above.

By the remarks of section 4.3, it turns out that Airy asymptotics, such as \tilde{f}^ϵ , are not adequate to describe the flow near “higher-order” caustics. More refined asymptotics of f^ϵ are required in these cases. Thus we may think of the Airy asymptotics of (5.1) or (5.2) as the first-order asymptotics that suffice for describing the field near fold or cusp caustics. In this sense, the limit Wigner function can be thought of as the zeroth order approximation, which recovers geometrical optics but fails near any caustic.

On the other hand, folds and cusps are the generic caustics in the one-dimensional case, in the sense that small perturbations of the Lagrangian manifold can destroy any caustic but them (cf. [BER2], [FLAN, Chap. 3]). From this point of view, the Airy asymptotics of f^ϵ provide the semiclassical Wigner function near the generic caustics.

Appendix A. The Airy-type uniform stationary phase expansion. In this appendix we recall the uniform stationary phase formula developed by Chester, Friedman and Ursell [CFU] (see also [BOR, sect. 2.3]). We consider the integral

$$(A.1) \quad I(\lambda, a) = \int_{-\infty}^{\infty} \exp(i\lambda\phi(x, a)) f(x) dx, \quad \alpha > 0,$$

with smooth f , for the case when the phase function $\phi \in C^\infty$ has two stationary points, $x_1(a)$ and $x_2(a)$, which approach the same limit x_0 when $a \rightarrow 0$. Let $\phi''_{xx}(x_1, a) < 0$ and $\phi''_{xx}(x_2, a) > 0$.

Assume also that

$$(A.2) \quad \phi'''_{xxx} \neq 0, \quad \phi'_x = \phi''_{xx} = 0, \quad \phi''_{xa} \neq 0$$

at $x = x_0, a = 0$. Then, for large λ ,

$$(A.3) \quad I(\lambda, a) = \exp(i\lambda\phi_0) \left(2A_0\pi\lambda^{-1/3} Ai(-\lambda^{2/3}\xi) - 2B_0\pi i\lambda^{-2/3} Ai'(-\lambda^{2/3}\xi) \right) + O(\lambda^{-4/3}),$$

where

$$(A.4a) \quad \begin{aligned} \phi_0 &= \frac{1}{2}(\phi(x_1, a) + \phi(x_2, a)) \\ &\approx \phi(x_0, 0) + O(a^2), \quad a \rightarrow 0, \end{aligned}$$

$$(A.4b) \quad \begin{aligned} \xi &= \left((3/4)(\phi(x_1, a) - \phi(x_2, a)) \right)^{2/3} \\ &\approx a\phi''_{xa}(x_0, 0) \left(\phi'''_{xxx}(x_0, 0)/2 \right)^{-1/3} + O(a^2), \quad a \rightarrow 0, \end{aligned}$$

$Ai(\cdot)$ denotes the Airy function, and the constants A_0, B_0 are given by

$$(A.5a) \quad A_0 = 2^{-1/2}\xi^{1/4} \left(\frac{f(x_2)}{\sqrt{|\phi''_{xx}(x_2, a)|}} + \frac{f(x_1)}{\sqrt{|\phi''_{xx}(x_1, a)|}} \right),$$

$$(A.5b) \quad B_0 = 2^{-1/2}\xi^{-1/4} \left(\frac{f(x_1)}{\sqrt{|\phi''_{xx}(x_1, a)|}} - \frac{f(x_2)}{\sqrt{|\phi''_{xx}(x_2, a)|}} \right).$$

Note that by using the asymptotics of the Airy function (see Appendix B), for fixed $a > 0$, as $\lambda \rightarrow \infty$, from (A.3) we obtain the formula

$$(A.6) \quad I(\lambda, a) = \sqrt{(2\pi)/\lambda} \left(\frac{f(x_2) \exp(i\lambda\phi(x_2, a) + i\pi/4)}{\sqrt{|\phi''_{xx}(x_2, a)|}} + \frac{f(x_1) \exp(i\lambda\phi(x_1, a) - i\pi/4)}{\sqrt{|\phi''_{xx}(x_1, a)|}} \right) + O(\lambda^{-1}), \quad \lambda \rightarrow \infty,$$

which is also obtained by summing the standard stationary phase contributions of the stationary points x_1 and x_2 , considered separately.

Appendix B. Properties of Airy functions. The Airy function is a C^∞ function defined by the integral [HO]

$$Ai(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left(i\left(\frac{\zeta^3}{3} + \zeta x\right)\right) d\zeta, \quad x \in \mathbb{R},$$

and it can be extended to an entire analytic function for $x \in \mathbb{C}$. It satisfies the Airy differential equation

$$A''(x) - xAi(x) = 0,$$

and for $x = 0$,

$$Ai(0) = 3^{-1/6}\Gamma(1/3)/2\pi = 3^{-2/3}/\Gamma(2/3).$$

The asymptotics of $Ai(x)$ for large $|x|$ are given by

$$Ai(x) \approx \frac{1}{2\sqrt{\pi}} x^{-1/4} e^{-2/3x^{3/2}}, \quad x \rightarrow +\infty,$$

$$Ai(-x) \approx \frac{1}{\sqrt{\pi}} x^{-1/4} \cos\left(\frac{2}{3}x^{3/2} - \frac{\pi}{4}\right), \quad x \rightarrow +\infty.$$

In what follows we will also use the formulas [OLV, pp. 434, 338]

$$\int_{-\infty}^{\infty} Ai(t) dt = 1,$$

$$\int_0^{\infty} Ai(t)t^{a-1} dt = \frac{\Gamma(a)}{3^{(a+2)/3}\Gamma((a+2)/3)}, \quad \Re(a) > 0.$$

B.1. Formulas related to the fold. We first derive the following “projection identity”:

$$(B.1) \quad \int_{-\infty}^{+\infty} Ai(ak^2 + bk + c) dk = \frac{2\pi}{\sqrt{a}} \frac{1}{2^{1/3}} Ai^2\left(-\frac{b^2 - 4ac}{4^{5/3}a}\right), \quad a > 0,$$

related to the amplitude of the solution at the fold. This has been derived by Berry [BER1, App. E] (see also [BEW] for similar identities corresponding to more general diffraction catastrophes). Changing variables by $u = \sqrt{ak} + b/(2\sqrt{a})$, we have

$$(B.2) \quad \int_{-\infty}^{+\infty} Ai(ak^2 + bk + c) dk = \frac{2}{\sqrt{a}} \int_0^{\infty} Ai(u^2 - \lambda) du,$$

where $\lambda = \frac{1}{4a}(b^2 - 4ac)$. Then we rewrite the right-hand side of (B.2) in the form

$$\int_0^\infty Ai(u^2 - \lambda)du = \frac{1}{2} \int_{-\infty}^\infty Ai(u^2 - \lambda)du = \frac{1}{4\pi} \int_{-\infty}^{+\infty} d\xi \int_{-\infty}^{+\infty} d\eta \exp\left(i\left(\frac{\eta^3}{3} + (\xi^2 - \lambda)\eta\right)\right),$$

and using the change of variables

$$\xi = (Y - X)/2^{\frac{2}{3}}, \quad \eta = (Y + X)/2^{\frac{2}{3}}$$

in the double integral, we obtain

$$\begin{aligned} & \frac{1}{4\pi} \int_{-\infty}^{+\infty} d\xi \int_{-\infty}^{+\infty} d\eta \exp\left(i\left(\frac{\eta^3}{3} + (\xi^2 - \lambda)\eta\right)\right) \\ (B.3) \quad &= \frac{1}{4\pi} \frac{1}{2^{1/3}} \int_{-\infty}^{+\infty} dX \int_{-\infty}^{+\infty} dY \exp\left(i\left(\frac{X^3}{3} + \frac{Y^3}{3} - \frac{\lambda}{2^{2/3}}(X + Y)\right)\right) \\ &= \pi \frac{1}{2^{1/3}} Ai^2\left(-\frac{\lambda}{2^{2/3}}\right), \end{aligned}$$

which gives (B.1).

We next prove the limit

$$(B.4) \quad \frac{1}{\sqrt{\eta}} Ai\left(\frac{y^2}{\eta}\right) \rightarrow \frac{\Gamma^2(1/3)}{2^{1/3}3^{1/3}2\pi} \delta(y), \quad \eta \rightarrow 0^+.$$

For some $\phi \in C_0^\infty(\mathbb{R})$, we have

$$\begin{aligned} & \int_{-\infty}^\infty \frac{1}{\sqrt{\eta}} Ai\left(\frac{y^2}{\eta}\right) \phi(y) dy = \int_0^\infty \frac{1}{\sqrt{\eta}} Ai\left(\frac{y^2}{\eta}\right) (\phi(y) + \phi(-y)) dy \\ &= \frac{1}{2} \int_0^\infty t^{-1/2} Ai(t) (\phi(\sqrt{\eta}t) + \phi(-\sqrt{\eta}t)) dt. \end{aligned}$$

As $\eta \rightarrow 0^+$, the last integral converges to

$$\phi(0) \int_0^\infty t^{-1/2} Ai(t) dt = \phi(0) \frac{\Gamma^2(1/3)}{2^{1/3}3^{1/3}2\pi},$$

which proves (B.4).

B.2. Formulas related to the cusp. Here we derive the “projection identity”

$$(B.5) \quad \int_{-\infty}^{+\infty} \frac{1}{|\xi|^{1/3}} Ai\left(\frac{\lambda}{\xi^{1/3}}(\xi^3 - v\xi + u)\right) d\xi = \frac{1}{2\pi} \frac{1}{\sqrt{2}\lambda^{1/4}} |P(-V, U)|^2, \quad \lambda > 0,$$

related to the amplitude of the wave field at the cusp. Here

$$(B.6a) \quad P(-V, U) = \int_{\mathbb{R}} \exp\left(i\left(\frac{t^4}{4} - V\frac{t^2}{2} + Ut\right)\right) dt$$

is the Pearcey integral, with arguments

$$(B.6b) \quad V = \frac{1}{\sqrt{2}} \lambda^{3/4} v, \quad U = \frac{1}{2^{3/4}} \lambda^{9/8} u.$$

By the integral representation of the Airy function, and using subsequently the transformations

$$\xi = \theta, \quad \rho = \theta^{1/3}\tau,$$

and

$$2^{3/4}\lambda^{3/8}\theta = t + s, \quad 2^{3/4}\lambda^{-1/8}\tau = t - s,$$

we have

(B.7)

$$\begin{aligned} & \int_{-\infty}^{+\infty} \frac{1}{|\xi|^{1/3}} Ai\left(\frac{\lambda}{\xi^{1/3}}(\xi^3 - v\xi + u)\right) d\xi \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \exp\left(i\frac{\rho^3}{3} + i\lambda\frac{\rho}{\xi^{1/3}}(\xi^3 - v\xi + u)\right) d\xi d\rho \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \exp\left(i\left(\frac{1}{3}\tau^3\theta + \lambda\tau\theta^3 - v\lambda\tau\theta + u\lambda\tau\right)\right) d\tau d\theta \\ &= \frac{1}{2\pi} \frac{1}{\sqrt{2}\lambda^{1/4}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \exp\left(i\left(\frac{t^4}{4} - V\frac{t^2}{2} + Ut\right)\right) \exp\left(-i\left(\frac{s^4}{4} - V\frac{s^2}{2} + Us\right)\right) dt ds \\ &= \frac{1}{2\pi} \frac{1}{\sqrt{2}\lambda^{1/4}} |P(-V, U)|^2. \end{aligned}$$

We close this appendix with the derivation of the limit

$$(B.8) \quad \frac{1}{\eta^{1/4}} \frac{1}{|y|^{1/3}} Ai\left(\frac{y^{8/3}}{\eta}\right) \rightarrow \frac{3^{1/4}2^{1/2}\Gamma^2(1/4)}{8\pi} \delta(y), \quad \eta \rightarrow 0^+.$$

For some $\phi \in C_0^\infty(\mathbb{R})$, we have

$$\frac{1}{\eta^{1/4}} \int_{-\infty}^{+\infty} \frac{1}{|y|^{1/3}} Ai\left(\frac{y^{8/3}}{\eta}\right) \phi(y) dy = \frac{1}{\eta^{1/4}} \int_0^{+\infty} \frac{1}{|y|^{1/3}} Ai\left(\frac{y^{8/3}}{\eta}\right) (\phi(y) + \phi(-y)) dy.$$

Changing variables to $t = y^{8/3}/\eta$, we see that the last integral, as $\eta \rightarrow 0^+$, converges to

$$\frac{3}{4} \int_0^{+\infty} t^{-3/4} Ai(t) dt \phi(0) = \frac{3^{1/4}2^{1/2}\Gamma^2(1/4)}{8\pi} \phi(0),$$

which proves (B.8).

Appendix C. Some L^2 -estimates. Here we present estimates of the L^2 -norms of W^ϵ , f^ϵ , and $W^\epsilon - f^\epsilon$, and in particular we will prove estimate (3.8) that we mentioned in section 3.1. We recall that we denote by f^ϵ the exact solution of the Wigner equation (3.1) and by W^ϵ the Wigner transform of the WKB solution (cf. (3.2)). All subsequent calculations are valid in $[0, T]$, $T < t_c$.

Assuming for simplicity that the amplitude $A(x, t)$ has initially ($t = 0$) compact support, it will keep having compact support for $0 \leq t \leq T$. We next compute the L^2 -norm of W^ϵ . Using (3.2), we have

$$\begin{aligned} \|W^\epsilon\|_{L^2(\mathbb{R}_x \times \mathbb{R}_k)}^2 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W^\epsilon(x, k, t) \overline{W^\epsilon(x, k, t)} dx dk \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\pi\epsilon} \int_{-\infty}^{\infty} D(\sigma; x, t) e^{i\frac{\epsilon}{\epsilon}F(\sigma; x, k, t)} d\sigma \frac{1}{\pi\epsilon} \int_{-\infty}^{\infty} D(\rho; x, t) e^{-i\frac{\epsilon}{\epsilon}F(\rho, x, k)} d\rho dx dk. \end{aligned}$$

Interchanging the order of integration and performing first the k -integration and then the ρ -integration, we find that

$$\|W^\epsilon\|_{L^2(\mathbb{R}_x \times \mathbb{R}_k)}^2 = \frac{1}{\pi\epsilon} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A^2(x + \sigma, t) A^2(x - \sigma, t) dx d\sigma.$$

Finally, changing variables by $x + \sigma = \xi$, $x - \sigma = \eta$, we obtain

$$(C.1) \quad \|W^\epsilon\|_{L^2(\mathbb{R}_x \times \mathbb{R}_k)}^2 = \frac{1}{2\pi\epsilon} \|A(\cdot, t)\|_{L^2(\mathbb{R})}^4 = \frac{1}{2\pi\epsilon} \|A_0\|_{L^2(\mathbb{R})}^4.$$

A similar calculation shows that

$$\|f^\epsilon\|_{L^2(\mathbb{R}_x \times \mathbb{R}_k)}^2 = \frac{1}{2\pi\epsilon} \|\psi_0^\epsilon\|_{L^2(\mathbb{R}_x)}^4 = \frac{1}{2\pi\epsilon} \|A_0\|_{L^2(\mathbb{R}_x)}^4$$

(cf. [LP, p. 605]). Thus the L^2 -norms of f^ϵ and W^ϵ are equal to each other and constant in time. Clearly, $f^\epsilon(x, k, t = 0) = W^\epsilon(x, k, t = 0)$. We next show that

$$(C.2) \quad \|f^\epsilon - W^\epsilon\|_{L^2(\mathbb{R}_x \times \mathbb{R}_k)} \leq C_T t \epsilon^{1/2}, \quad 0 \leq t \leq T.$$

Since the main ideas are in [LP], we will mainly sketch the proof of it. Setting $g^\epsilon = f^\epsilon - W^\epsilon$ and plugging g^ϵ into the Wigner equation (2.7), we have by (3.7) that

$$(C.3) \quad L^\epsilon[g^\epsilon] = -\frac{i}{2\pi} I_\epsilon(x, k, t), \quad g^\epsilon|_{t=0} = 0.$$

Recall that

$$I_\epsilon(x, k, t) = \int e^{\frac{i}{\epsilon} F} (A(x + \sigma, t) A(x - \sigma, t))_{x\sigma} d\sigma.$$

A calculation similar to the one leading to (C.1) yields

$$(C.4) \quad \|I_\epsilon\|_{L^2(\mathbb{R}_x \times \mathbb{R}_k)}^2 = \pi\epsilon \left(\|A_{xx}\|_{L^2}^2 \|A\|_{L^2}^2 - \|A_x\|_{L^2}^4 \right) \leq 16\pi^2 C_T^2 \epsilon,$$

where $C_T^2 = \frac{1}{16\pi} \sup_{\{0 \leq t \leq T\}} (\|A_{xx}\|_{L^2}^2 \|A\|_{L^2}^2 - \|A_x\|_{L^2}^4)$. Multiplying (C.3) by g^ϵ and integrating over $\mathbb{R}_x \times \mathbb{R}_k$, we end up with (cf. [LP, p. 607])

$$\frac{d}{dt} \|g^\epsilon\|_{L^2}^2 \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I_\epsilon(x, k, t) g^\epsilon(x, k, t) dx dk \leq \frac{1}{2\pi} \|g^\epsilon\|_{L^2} \|I_\epsilon\|_{L^2}.$$

From this and (C.4) we get that

$$\frac{d}{dt} \|g^\epsilon\|_{L^2} \leq C_T \epsilon^{1/2}, \quad 0 \leq t \leq T,$$

from which (C.2) follows upon integration.

Appendix D. Two-phase geometrical optics. Near the fold, classical geometric optics predicts the field [BB], [LU], [KR]

$$(D.1) \quad \psi^\epsilon(x, t) = A_+(x, t) \exp\left(\frac{i}{\epsilon} S^+(x, t)\right) \exp\left(\frac{i\pi}{2}\right) + A_-(x, t) \exp\left(\frac{i}{\epsilon} S^-(x, t)\right),$$

Here $A_\pm(x, t) = \frac{A_0(r_\pm)}{\sqrt{|J^\pm(x, t)|}}$, with r_\pm the points from where the two rays arriving at (x, t) emanate, J^\pm the Jacobian and $S^\pm(x, t)$ the solutions of the Hamilton–Jacobi

equation (1.7) computed along these two rays. Note that the superscript $-/+$ corresponds to the ray which arrives at (x, t) before/after hitting the caustic, respectively, and the factor $\exp(i\pi/2)$ takes account of the phase shift as the second ray passes from the caustic. Then, with $A_{\pm}(x, t) = A_{\pm}$, we have

$$(D.2) \quad |\psi^{\epsilon}(x, t)|^2 = A_+^2 + A_-^2 + 2A_+A_- \sin\left(\frac{1}{\epsilon}(S^+(x, t) - S^-(x, t))\right).$$

The last term in (D.2) is (classically) negligible as $\epsilon \rightarrow 0$, thanks to the fact that $x^m \exp(\frac{ix}{\epsilon}) \rightarrow 0$, as $\epsilon \rightarrow 0$, for any $m \geq 0$, as a generalized function in \mathcal{D}' , and therefore

$$(D.3) \quad |\psi^{\epsilon}(x, t)|^2 \rightarrow A_+^2(x, t) + A_-^2(x, t), \quad \epsilon \rightarrow 0.$$

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