REFINED ASYMPTOTICS FOR THE BLOWUP OF $u_t - \Delta u = u^p$.

By

Stathis Filippas

and

Robert V. Kohn

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Stathis Filippas[1]
Institute for Mathematics and Its Applications
University of Minnesota

and

Robert V. Kohn[2]
Courant Institute
New York University

Abstract

This work is concerned with positive, blowing-up solutions of the semilinear heat equation $u_t - \Delta u = u^p$ in R^n. Our main contribution is a sort of center manifold analysis for the equation in similarity variables, leading to refined asymptotics for u in a backward space-time parabola near any blowup point. We also explore a connection between the asymptotics of u and the local geometry of the blowup set.

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1 Introduction.

This work is concerned with positive, blowing-up solutions of the semilinear heat equation

$$u_t - \Delta u = u^p \quad in \quad R^n \times (0,T), \quad p > 1. \tag{1.1}$$

Our main contribution is a sort of center manifold analysis, leading to refined asymptotics for u in a backward space-time parabola near any blowup point. We also explore an apparent connection between the asymptotics of u and the local geometry of the blowup set.

A lot of work has been done concerning the blowup of solutions of (1.1), and related equations such as $u_t - \Delta u = e^u$. A comprehensive review is beyond the scope of this introduction; discussions and biblographies will be found in [1,2,8,9,11-13].

To explain the relevance of center manifold theory, let us briefly review the theory developed by Giga and Kohn based on *similarity variables.* This change of both dependent and independent variables is defined by

$$\begin{aligned} w(y,s) &= (T-t)^{\frac{1}{p-1}} u(x,t) \\ y = (x-a)/\sqrt{T-t}, &\quad s = -ln(T-t), \end{aligned} \tag{1.2}$$

where a is a blowup point and T the blowup time. If u solves (1.1) then w solves

$$w_s - \frac{1}{\rho}\nabla\cdot(\rho\nabla w) + \frac{1}{p-1}w = w^p, \tag{1.3}$$

where we have set $\rho(y) = e^{-|y|^2/4}$. Studying the behavior of u near blowup is equivalent to studying the large-time behavior of w. Let us assume that u is nonnegative and p is "subcritical," i.e. $n = 1,2$ or, if $n \geq 3$, $p < (n+2)/(n-2)$. Then it follows from [12,13] that

$$w(y,s) \to \kappa \qquad \text{as} \qquad s \to \infty, \tag{1.4}$$

uniformly on bounded sets $|y| \leq C$, where κ is the constant, stationary solution of (1.3):

$$\kappa = (p-1)^{-1/(p-1)}. \tag{1.5}$$

To learn more about how w approaches its limit κ, it is natural to linearize (1.3) about this critical point. Setting $w = \kappa + v$, and using Taylor expansion to get

$$w^p - \frac{1}{p-1}w = v + \frac{p}{2\kappa}v^2 + \dots,$$

one can write (1.3) as

$$v_s - \frac{1}{\rho}\nabla\cdot(\rho\nabla v) - v = \frac{p}{2\kappa}v^2 + O(v^3). \tag{1.6}$$

Ordinarily quadratic and higher-order terms are ignored in the course of "linearization." However, in this case the critical point $w \equiv \kappa$ is not hyperbolic; in other words, the linear operator $\frac{1}{\rho}\nabla\cdot(\rho\nabla v) + v$ has a nontrivial null-space. Therefore the quadratic term in (1.6) cannot be ignored. The role of center manifold theory should now be clear: it is the standard tool (at least in finite dimensions) for studying the solutions of a nonlinear evolution equation in a neighborhood of a critical point which is not hyperbolic, see e.g. [6, 15, 20].

At the formal level, a center manifold analysis of (1.6) is relatively easy. This might also be called a "weakly nonlinear stability analysis." We develop it in Section 2, focussing for simplicity on space dimensions 1 and 2. The formal argument suggests that generically

$$v(y,s) \sim \frac{\kappa}{2ps}(1-\frac{1}{2}y^2) \quad \text{in } R^1 \tag{1.7}$$

$$v(y,s) \sim \frac{\kappa}{ps}(1-\frac{1}{4}|y|^2) \quad \text{in } R^2 . \tag{1.8}$$

It also indicates that the blowup set should generically consist of isolated points. Blowup along a one-dimensional continuum in R^2 is known to be possible [13]; an asymptotic law different from (1.8) applies in this case, see (2.33).

The remaining sections 3-8 are fully rigorous. Their goal is to justify, in so far as possible, the picture developed in Section 2. This is not an easy task: the methods usually used to justify center manifold theory in an infinite dimensional setting do not apply in the present context. (See Section 3 for an explanation why not.) Basically, we get around this problem by using the special structure of the equation (1.3).

Our rigorous results apply to any nonnegative solution of the Cauchy problem (1.1) satisfying

$$\| u \|_{L^\infty}(t) \le C(T-t)^{-1/p-1} \tag{1.9}$$

and having the asymptotic behavior

$$(T-t)^{1/(p-1)}u(a+y\sqrt{T-t},t) \to \kappa \text{ as } t\uparrow T, \text{ uniformly for } |y|\le C. \tag{1.10}$$

To state them, we must first introduce some notation. Let L^2_ρ be the space of functions $v(y)$ such that $\int v^2\rho dy < \infty$, where $\rho = e^{-|y|^2/4}$ as usual. This is a Hilbert space with inner product $(u,v) = \int uv\rho dy$. The operator

$$\mathcal{L}v = \frac{1}{\rho}\nabla\cdot(\rho\nabla v)+v \tag{1.11}$$

is self-adjoint, with eigenvalues $1, 1/2, 0, -1/2, -1, \ldots$. Let $\{e^+\}_{j=1}^k$ be its eigenfunctions with positive eigenvalues; $\{e_j^0\}_{j=1}^m$ its eigenfunctions with eigenvalue zero; and $\{e_j^-\}_{j=1}^\infty$ its eigenfunctions with negative eigenvalues. (We shall prove in Section 2 that $k=n+1$ and $m=n(n+1)/2$.) We may then expand

$$v(y,s) = \sum_{j=1}^k \beta_j(s)e_j^+(y) + \sum_{j=1}^m \alpha_j(s)e_j^0(y) + \sum_{j=1}^\infty \gamma_j(s)e_j^-(y), \tag{1.12}$$

where $v(y,s)$ is related to $u(x,t)$ as in (1.2)-(1.6).

Our first main result asserts that the neutral modes are dominant as $s\to\infty$, unless v tends to zero exponentially fast:

Theorem A. *Either $v\to 0$ exponentially fast as $s\to\infty$, or else for any $\varepsilon>0$ there is a time s_0 such that*

$$\sum_{j=1}^k \beta_j^2 + \sum_{j=1}^k \gamma_j^2(s) \le \varepsilon \sum_{j=1}^m \alpha_j^2(s) \quad for \quad s\ge s_0. \tag{1.13}$$

We believe that the situation $v \to 0$ exponentially fast is in some sense exceptional, and the behavior (1.13) generic. This would be the case in finite dimensions, but we are unable to prove it in the present context.

Our second main result serves to justify the "equation on the center manifold" which is the essence of the formal analysis:

Theorem B. *Assume that v does not approach 0 exponentially fast. Then the neutral modes $\{\alpha_j\}_{j=1}^m$ satisfy*

$$\dot{\alpha}_j = \frac{p}{2\kappa}\pi_j^0(v_0^2) + O\left(\varepsilon\sum_{j=1}^m \alpha_j^2\right), \tag{1.14}$$

where π_j^0 denotes orthogonal projection onto e_j^0 and v_0 is the neutral component of v,

$$v_0 = \sum_{j=1}^m \alpha_j(s)e_j^0(y). \tag{1.15}$$

The error term on the right in (1.14) is small compared to the other one. If we neglect it, then (1.14) amounts to an $m \times m$ system of nonlinear ODE's for the functions $\alpha_j(s)$. We shall make this system more explicit and study its behavior in Section 2.

Our third main result gives refined asymptotics for the blowup of u, in one space dimension:

Theorem C. *Assume that the spatial dimension is $n = 1$, and that v does not approach 0 exponentially fast. Then for any $C > 0$ and $\varepsilon > 0$ there exists s_0 such that*

$$sup_{|y|<C}|v(y,s) - \frac{\kappa}{2ps}(1 - \frac{1}{2}y^2)| = O(\frac{\varepsilon}{s}) \tag{1.16}$$

when $s \geq s_0$.

Of course, refined asymptotics for v are equivalent to refined asymptotics for u. Restated in terms of u, (1.16) becomes

$$(T-t)^{1/p-1}\, u(x,t) \sim \kappa + \frac{\kappa}{2p|ln(T-t)|}\left(1 - \frac{|x-a|^2}{2(T-t)}\right), \tag{1.17}$$

in the sense that the difference is $o(|ln(T-t)|^{-1})$ as $t \uparrow T$, uniformly in parabolas $|x-a|^2 \leq C(T-t)$.

Our fourth main result concerns the connection between the asymptotics of v and the character of the blowup set. We shall prove

Theorem D. *Assume that the spatial dimension is $n = 1$ and that v does not approach 0 exponentially fast . Then the center of scaling is an isolated blowup point.*

Chen and Matano have shown that in one space dimension blowup always occurs at isolated points [5]. Theorem D is weaker than their result, since it has the additional hypothesis that v does not approach 0 exponentially fast. However, the argument in [5] is intrinsically one-dimensional. Our Theorem D, by contrast, has a natural extension to higher dimensions. See the end of Section 7 for further discussion.

After this work was completed we learned that a formal analysis similar to that of Section 2 has been done simultaneously and independently by Galaktionov, Herrero, and Velazquez [10]. In addition, rigorous results similar to our Theorems A-D have been proved simultaneously and independently (but only in one space dimension) by Herrero and Velazquez [16-18]. Curiously, at the technical level our analysis is quite different from that of [16-18]. The results of Herrero and Velazquez are in some ways more comprehensive than ours: they get refined asymptotics for $|y|^2 \sim s$ as well as for $|y|^2 \leq C$, and they consider what happens if $v \to 0$ exponentially fast. On the other hand, they make extensive use of the maximum principle, whereas we rely instead on weighted energy estimates, so our approach might extend more easily to systems.

Our results have recently been exploited by Liu [21]. His work includes the analogue of Theorem C for radial solutions in R^n, $n \geq 2$. It also includes a proof that the solution of (1.14) satisfies $|\alpha(s)| \sim \frac{1}{s}$ when $n = 2$.

Center manifold analysis has also played a major role in the recent work of Bressan on $u_t - \Delta u = e^u$ [3,4]. (We presume that something like his analysis could be done for $u_t - \Delta u = u^p$; the analogue of the present work in one space dimension for the exponential nonlinearity is included in [16].) His approach is quite different from ours, however. For one thing, Bressan works with $z = y/\sqrt{s}$ rather than y as the spatial variable; thus he naturally obtains asymptotics valid in the larger region $y^2 \leq Cs$ rather than $|y| \leq C$. In addition, the focus of his work is to show the *existence* of solutions with certain asymptotics, while our goal and that of [16-18] is to describe the asymptotic behavior of *any* solution. We wonder whether there might be some way of combining the strengths of both approaches.

Yet another method for analyzing the asymptotics of blowup has recently been explored by Keller and Lowengrub [19]. They use a change of variable totally different from (1.2): if u solves (1.1) they work with $U = u^{-(p-1)}$, which tends to zero linearly as $t \uparrow T$. This is useful both for formal asymptotics and for the numerical calculation of u. We are unaware of any direct connection between their approach and the others discussed above.

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2 A Formal Analysis.

We have explained that the behavior of u near blowup is encoded in the large-time behavior of $v(y,s)$. So the essential question is this: given a solution of (1.6) which tends to zero as $s \to \infty$, what is the asymptotic profile of v ? If (1.6) were a finite dimensional dynamical system, then answering this question would be a fairly routine matter. The picture that emerges is presented in this section. Our treatment is "formal" in the sense that we cannot prove its validity for (1.6); however most of what we do would be rigorously correct for an analogous finite-dimensional system. (Some of our conclusions will be justified later in Theorems A-D.)

2.1 The Linear Operator.

Our first task is to understand the linear operator $\mathcal{L}$, defined by (1.11), viewed as a self-adjoint operator on L^2_ρ. It is convenient to treat the one-dimensional case first.

Lemma 2.1 *In R^1 the eigenvalues of $\mathcal{L}$ are*

$$\lambda_k = \frac{-k}{2} + 1, \; k = 0, 1, 2, \ldots . \tag{2.1}$$

The associated orthonormal eigenfunctions are

$$h_k(y) = \alpha_k H_k(y/2) \tag{2.2}$$

where H_k is the k^{th} Hermite polynomial and α_k is a normalizing factor. The first few are

$$h_0 = c_0, \; h_1 = c_1 y, \; h_2 = c_2(\frac{1}{2}y^2 - 1) \tag{2.3}$$

with $c_1 = c_2 = \frac{1}{2}\pi^{-1/4}$ and $c_0 = \frac{1}{\sqrt{2}}\pi^{-1/4}$.

Proof: In one space dimension our operator is

$$\mathcal{L}v = v_{yy} - \frac{y}{2}v_y + v.$$

Changing variables to $x = y/2$ and $\phi(x) = v(y)$, we see that $\mathcal{L}v = \lambda v$ if and only if

$$\phi_{xx} - 2x\phi_x = 4(\lambda - 1)\phi. \tag{2.4}$$

It is well-known that the eigenfunctions of (2.4) are the Hermite polynomials, defined by

$$H_k(x) = (-1)^k e^{x^2} \frac{d^k}{dx^k}(e^{-x^2}), \; k = 0, 1, 2, \ldots, \tag{2.5}$$

see e.g. [22]. The eigenvalue associated to $\phi = H_k$ is $4(\lambda_k - 1) = -2k$. Changing variables back to y, this leads to (2.1)-(2.2).

The Hermite polynomials are known to form an orthogonal basis for $L^2_{\gamma(x)}$ with weight $\gamma(x) = e^{-x^2}$. So $\{h_k(y)\}_{k=0}^\infty$ form an orthonormal basis for L^2_ρ.

The values of the first few normalizing constants can easily be calculated by evaluating the integrals $\int \rho dy$, $\int y^2 \rho dy$, etc., leading to (2.3). Alternatively, they can be deduced from the general rule

$$\alpha_k = (\pi^{1/2} 2^{k+1} k!)^{-1/2}, \tag{2.6}$$

which follows from the formula for the normalizing constants of the Hermite polynomials.

□

We note for future reference that

$$\frac{d}{dy}h_k(y) = (\frac{k}{2})^{1/2} h_{k-1}(y), \quad k = 1, 2, \ldots . \tag{2.7}$$

This is a consequence of the well-known recursion formula $H_k' = 2kH_{k-1}$, using (2.6) and change of variables.

In the multidimensional case, the eigenfunctions of $\mathcal{L}$ are products of Hermite polynomials.

Lemma 2.2 *In $R^n, n \geq 2$, the eigenvalues of $\mathcal{L}$ are still given by (2.1). The corresponding normalized eigenfunctions are as follows:*

$$\begin{array}{lll} \textit{for } \lambda_0 = 1, & h_0^n & \\ \textit{for } \lambda_1 = 1/2, & h_0^{n-1} h_1(y_i) & i = 1, 2, \dots, n \\ \textit{for } \lambda_2 = 0, & h_0^{n-1} h_2(y_i) & i = 1, 2, \dots, n \\ & h_0^{n-2} h_1(y_i) h_1(y_j), & i \neq j,\ i, j = 1, \dots, n \end{array}$$

and so forth. In particular, the null space of $\mathcal{L}$ has dimension $n + \begin{pmatrix} n \\ 2 \end{pmatrix} = \frac{n(n+1)}{2}$. In the special case $n = 2$, the neutral eigenfunctions are

$$c_0 c_2 (\frac{1}{2} y_1^2 - 1),\ c_0 c_2 (\frac{1}{2} y_2^2 - 1),\ c_1^2 y_1 y_2, \tag{2.8}$$

where c_i are as in (2.3).

Proof. In one space dimension we have shown that

$$\frac{1}{\rho} \nabla \cdot (\rho \nabla h_k) = \frac{-k}{2} h_k.$$

It follows easily that in several dimensions $h = h_{k_1}(y_1) \dots h_{k_n}(y_n)$ satisfies

$$\frac{1}{\rho} \nabla \cdot (\rho \nabla h) = -\frac{1}{2} (\sum_{i=1}^{n} k_i) h\,.$$

Thus h is an eigenfunction of $\mathcal{L}$ with eigenvalue $1 - \frac{1}{2} \sum k_i$. We note moreover that h is normalized. This construction gives all the eigenvalues and eigenfunctions, since the products $h_{k_1}(y_1) \dots h_{k_n}(y_n)$ are easily seen to be dense in L_ρ^2.

The final assertion (2.8) follows from the more general one by means of (2.3).

□

2.2 Space Dimension One.

When the spatial dimension is $n = 1$ the eigenspaces of $\mathcal{L}$ are one-dimensional. This makes the analysis somewhat simpler than in higher dimensions.

Consider a solution $v = v(y, s)$ of (1.6) which tends to zero as $s \to \infty$. We can deduce the generic profile of v as follows. First, we drop term of order v^3 in the equation, since it is small compared to v^2 ; our attention is thus on the equation

$$v_s - \mathcal{L} v = \frac{p}{2\kappa} v^2. \tag{2.9}$$

Next, we decompose $v(y, s)$ into its various "modes," using the eigenfunctions of $\mathcal{L}$:

$$v = [\beta_1(s) h_0(y) + \beta_2(s) h_1(y)] + [\alpha_1(s) h_2(y)] + [\gamma_1(s) h_3(y) + \gamma_2(s) h_4(y) + \dots]. \tag{2.10}$$

Our notation here is consistent with (1.12): β_1, β_2 correspond to the "unstable" modes of $\mathcal{L}$, α_1 to the "neutral" one, and $\gamma_1, \gamma_2, \dots$ to the "stable" modes.

We assert that the "stable" and "unstable" modes are typically negligible in magnitude, i.e. that v is well-approximated by the ansatz

$$v(y,s) \sim \alpha_1(s)h_2(y). \tag{2.11}$$

Indeed, if $\beta_1(s)$ or $\beta_2(s)$ were significantly different from zero they would quickly dominate (since they grow exponentially), contradicting the assumption that $v \to 0$ as $s \to \infty$. If γ_i were initially large, it would decay exponentially, eventually becoming small compared with $\alpha_1(s)$ – which, as we shall see, decays algebraically. (A more analytical justification of (2.11) will be given below, based on center manifold theory).

If one accepts the ansatz (2.11) then the rest is easy. To get an ODE for $\alpha_1(s)$, we substitute (2.11) into the equation (2.9), then project the result in the direction $h_2(y)$. This yields

$$\dot{\alpha}_1(s) = \frac{p}{2\kappa}\alpha_1^2(s)\int h_2^3\rho dy.$$

The integral is easily computed explicitly, using the relations $h_2\rho = -(\rho h_1)'$ and $h_2' = h_1$ and integration by parts. The result is

$$\int h_2^3\rho dy = 4c_2$$

with c_2 as in (2.3). Thus $\alpha_1(s)$ solves

$$\dot{\alpha}_1 = \frac{2pc_2}{\kappa}\alpha_1^2. \tag{2.12}$$

The solution is

$$\alpha_1(s) = \left[\alpha_1^{-1}(s_0) - \frac{2pc_2}{\kappa}(s-s_0)\right]^{-1}, \tag{2.13}$$

in terms of the value at any fixed time s_0. We note that $\alpha_1^{-1}(s_0)$ must be negative, since otherwise (2.13) would blowup in finite time. We derived (2.13) from (2.11), which is of course only approximately valid. So we do not expect (2.13) to be exactly correct. It should, however, give the correct large-time asymptotics. From (2.13) we get

$$\alpha_1(s) = \frac{-\kappa}{2pc_2 s} + O(\frac{1}{s^2}); \tag{2.14}$$

combining this with (2.11) we deduce that

$$v(y,s) \sim \frac{\kappa}{2ps}(1 - \frac{1}{2}y^2). \tag{2.15}$$

We have thus given a formal justification for (1.7).

Now we explain how the preceding could be made rigorous in a finite-dimensional setting. There would exist a codimension two, locally defined invariant manifold, the *center-stable manifold*. It would be tangent at 0 to the span of the neutral and stable eigenfunctions of $\mathcal{L}$, and it would contain any trajectory that tends to 0 as $s \to \infty$. In particular, $\beta = (\beta_1, \beta_2)$ in (2.10) would be functions of α_1 and $\gamma = (\gamma_1, \gamma_2, \ldots)$ with $|\beta| \le C(|\alpha_1|^2 +$

$|\gamma|^2$). Within the center-stable manifold there would exist a one-dimensional, locally defined invariant manifold, the *center manifold.* It would be tangent at 0 to the span of the neutral eigenfunction h_2, and it would attract exponentially any trajectory that tends to 0 as $s \to \infty$. Hence generically solutions would tend to 0 like those on the center manifold. If v is on the center manifold then $\gamma = (\gamma_1, \gamma_2, \ldots)$ and $\beta = (\beta_1, \beta_2)$ are determined by α_1, with $|\beta| + |\gamma| \leq C|\alpha_1|^2$. In particular, $v(y, s) = \alpha_1(s)h_2(y) + O(\alpha_1^2)$, justifying (2.11). This would imply

$$\dot{\alpha}_1 = \frac{2pc_2}{\kappa}\alpha_1^2 + O(\alpha_1^3),$$

which would lead once again to (2.15). This discussion amounts to a summary of center manifold theory; detailed accounts can be found, for example, in [6,15,20].

We are interested only in solutions v of (1.6) which tend to zero as $s \to \infty$. Accordingly, the initial data of v are not arbitrary; rather, they should lie on the center-stable manifold. This reflects the fact that we fixed two parameters in arriving at (1.6): the blowup time T and the blowup point a.

The ansatz (2.11) is generically accurate, but there should be exceptional solutions which behave differently. In the context of center manifold theory these are the solutions on the *stable manifold.* They should approach 0 exponentially fast. These are solutions in which the neutral mode is negligible; they are described by the ansatz

$$v(y, s) \sim \gamma(s)h_l(y)\ ,\ l > 2. \tag{2.16}$$

If we assume that v is an even function of y then only even l can arise in (2.16). (It is proved in [17] that l must be even, without any symmetry hypothesis on v.) Only one mode appears in (2.16) because the eigenspaces of $\mathcal{L}$ are simple: modes higher than l decay faster, and so are eventually negligible, while modes lower than l are absent by hypothesis if (2.16) applies. If for example $l = 4$ then

$$v(y, s) \sim \gamma(s)h_4(y) = c_4\gamma(s)\left(\frac{y^4}{12} - y^2 + 1\right),$$

with $\gamma(s)$ approaching 0 exponentially fast. This corresponds to a solution of the u equation with *two local maxima which coalesce at blowup.*

2.3 Space Dimension Two.

The logic of the preceding discussion applies in any space dimension. However, when $n \geq 2$ more analysis is required to determine the generic behavior, since the eigenspaces of $\mathcal{L}$ are multidimensional. We shall discuss only the case $n = 2$. The neutral eigenfunctions are then given by (2.8); we label them as follows:

$$e_1^0(y) = c_0c_2(\frac{1}{2}y_1^2 - 1), \quad e_2^0(y) = c_0c_2(\frac{1}{2}y_2^2 - 1), \quad e_3^0(y) = c_1^2y_1y_2. \tag{2.17}$$

The analogue of (2.11) is

$$v(y, s) \sim \alpha_1(s)e_1^0(y) + \alpha_2(s)e_2^0(y) + \alpha_3(s)e_3^0(y). \tag{2.18}$$

As before, we determine a system of ODE's for $\alpha_i(s)$ by substituting (2.18) into the equation (2.9) then projecting the result onto e_j^0, $j = 1, 2, 3$. One computes that

$$\int (e_1^0)^3 \rho dy = \int (e_2^0)^3 \rho dy = 4c_2c_0$$

$$\int (e_3^0)^2 e_1^0 \rho dy = \int (e_3^0)^2 e_2^0 \rho dy = 2c_2c_0$$

with c_i as in (2.3). All other integrals of the form $\int e_i^0 e_j^0 e_k^0 \rho dy$ vanish. We therefore arrive at the system

$$\begin{aligned} \dot{\alpha}_1 &= \frac{p}{2\kappa}(4c_2c_0\alpha_1^2 + 2c_2c_0\alpha_3^2) \\ \dot{\alpha}_2 &= \frac{p}{2\kappa}(4c_2c_0\alpha_2^2 + 2c_2c_0\alpha_3^2) \\ \dot{\alpha}_3 &= \frac{p}{2\kappa}(4c_2c_0\alpha_1\alpha_3 + 4c_2c_0\alpha_2\alpha_3). \end{aligned} \tag{2.19}$$

To determine the asymptotic profile of v, it remains to study how solutions of (2.19) behave as they approach 0. We do this in the following

Lemma 2.3 *Assume that α is a solution of (2.19) which is not identically zero and which satisfies $\alpha(s) \to 0$ as $s \to \infty$. Then there are two possibilities. Either $\alpha_3^2 - 2\alpha_1\alpha_2$ never vanishes, in which case*

$$\begin{aligned} \alpha_1(s) &= \frac{-\kappa}{2c_2c_0p}\frac{1}{s} + O(s^{-2}) \\ \alpha_2(s) &= \frac{-\kappa}{2c_2c_0p}\frac{1}{s} + O(s^{-2}) \\ \alpha_3(s) &= O(s^{-2}) \quad ; \end{aligned} \tag{2.20}$$

or else $\alpha_3^2 - 2\alpha_1\alpha_2$ vanishes identically, in which case

$$\begin{aligned} \alpha_1(s) &= \frac{-\eta_1^2\kappa}{2c_2c_0p}\ \frac{1}{s}\ +\ O(s^{-2}) \\ \alpha_2(s) &= \frac{-\eta_2^2\kappa}{2c_2c_0p}\ \frac{1}{s}\ +\ O(s^{-2}) \\ \alpha_3(s) &= \frac{-\eta_1\eta_2\kappa}{\sqrt{2}c_2c_0p}\ \frac{1}{s}\ +\ O(s^{-2}) \end{aligned} \tag{2.21}$$

for some unit vector $\eta = (\eta_1, \eta_2)$ in R^2.

Proof: As a first step, we observe that $\alpha_1 \leq 0$, $\alpha_2 \leq 0$, and $\alpha_1 + \alpha_2 < 0$ for all s. Indeed, if $\alpha_1(s_0) > 0$ for some s_0 then (2.19) would force α_1 to blowup in finite time, contradicting the hypothesis that $\alpha(s)$ exists for all $s \to \infty$. This shows that $\alpha_1 \leq 0$, and the same argument applies to α_2. If $\alpha_1(s_0) + \alpha_2(s_0) = 0$ then $\alpha_1(s_0) = \alpha_2(s_0) = 0$, and it follows from (2.19) that $\alpha(s) \equiv 0$. Since by hypothesis α is not identically zero, we conclude that $\alpha_1 + \alpha_2 < 0$.

Next we make the change of dependent variable

$$X = \alpha_3^2 - 2\alpha_1\alpha_2 \,,\ Y = \alpha_1 - \alpha_2 \,,\ Z = \alpha_1 + \alpha_2. \tag{2.22}$$

A straightforward calculation using (2.19) and (2.22) gives

$$\begin{aligned} \dot{X} &= \gamma X Z \\ \dot{Y} &= \gamma Y Z \\ \dot{Z} &= \gamma \left(Z^2 + X\right) \end{aligned} \tag{2.23}$$

with $\gamma = 2c_2c_0p/\kappa$. Since $Z \neq 0$, it follows from (2.23) that

$$\begin{aligned} \frac{d}{ds}\left(\frac{X}{Z}\right) &= -\gamma\left(\frac{X}{Z}\right)^2 \\ \frac{d}{ds}\left(\frac{Y}{Z}\right) &= -\gamma\left(\frac{Y}{Z}\right)\cdot\left(\frac{X}{Z}\right). \end{aligned} \tag{2.24}$$

Solving the first of these equations, we find that either

$$X \equiv 0 \tag{2.25}$$

or else

$$\frac{X}{Z} = (\gamma s + \mu_0)^{-1} \tag{2.26}$$

where μ_0 is an undetermined constant.

Consider first (2.26). This is the case when $X = \alpha_3^2 - 2\alpha_1\alpha_2$ never vanishes. Solving (2.24) yields

$$\frac{Y}{Z} = \frac{\mu_1}{\gamma s + \mu_0} \tag{2.27}$$

where μ_1 is another constant of integration. To determine Z, we combine (2.26) and (2.23) to get

$$\dot{Z} = \gamma\left(Z^2 + \frac{Z}{\gamma s + \mu_0}\right).$$

The solution is

$$Z = \frac{-(\gamma s + \mu_0)}{\frac{1}{2}(\gamma s + \mu_0)^2 + \mu_2}, \tag{2.28}$$

where μ_2 is a constant of integration. Since $\alpha_1 = (Z+Y)/2$, $\alpha_2 = (Z-Y)/2$, it follows from (2.27) and (2.28) that

$$\alpha_1 = -\gamma^{-1}s^{-1} + O(s^{-2}),\ \alpha_2 = -\gamma^{-1}s^{-1} + O(s^{-2})$$

Since $\alpha_3^2 = X + 2\alpha_1\alpha_2 = X + \frac{1}{2}(Z^2 - Y^2)$, we also get after some calculation that

$$\alpha_3^2 = O(s^{-4}).$$

Thus if (2.26) holds then the large-time asymptotics of α are determined by (2.20).

Now consider the other case (2.25), when $X = \alpha_3^2 - 2\alpha_1\alpha_2 \equiv 0$. By (2.23) we have

$$\alpha_1 + \alpha_2 = Z = \frac{-1}{(\gamma s + \mu_0)} \tag{2.29}$$

for some μ_0. By (2.24) the ratio Y/Z is constant; we denote its value by μ_1. Evidently

$$\frac{\alpha_1}{\alpha_1+\alpha_2} - \frac{\alpha_2}{\alpha_1+\alpha_2} = \mu_1$$
$$\frac{\alpha_1}{\alpha_1+\alpha_2} + \frac{\alpha_2}{\alpha_1+\alpha_2} = 1,$$

so the ratios $\alpha_1/(\alpha_1+\alpha_2)$ and $\alpha_2/(\alpha_1+\alpha_2)$ are both constant. Since they lie between 0 and 1, we may choose $\eta = (\eta_1, \eta_2)$ such that $|\eta|^2 = 1$ and

$$\frac{\alpha_1}{\alpha_1+\alpha_2} = \eta_1^2, \; \frac{\alpha_2}{\alpha_1+\alpha_2} = \eta_2^2. \tag{2.30}$$

Notice that η_1 and η_2 are only determined up to a factor of ± 1. Since $\alpha_3^2 = 2\alpha_1\alpha_2$, we may choose the signs so that

$$\alpha_3 = \sqrt{2}\eta_1\eta_2(\alpha_1 + \alpha_2). \tag{2.31}$$

Then (2.29)-(2.31) yield

$$\dot{\alpha_1} = -\frac{\eta_1^2}{\gamma}\frac{1}{s} + O(s^{-2})$$
$$\dot{\alpha_2} = -\frac{\eta_2^2}{\gamma}\frac{1}{s} + O(s^{-2})$$
$$\dot{\alpha_3} = -\frac{\sqrt{2}\eta_1\eta_2}{\gamma}\frac{1}{s} + O(s^{-2}).$$

Since $\gamma = 2c_2c_0p/\kappa$, this is the same as (2.21).

□

It is now a simple matter to deduce the asymptotic profile of v : we need only substitute the behavior of α into (2.17)-(2.18). The generic case is (2.20), since it applies whenever $\alpha_3^2 \neq 2\alpha_1\alpha_2$. It yields

$$v(y,s) \sim \frac{\kappa}{ps}\left(1 - \frac{1}{4}|y|^2\right), \tag{2.32}$$

which is the same as (1.8). In this case v is asymptotically radially symmetric with an isolated maximum at $y = 0$. This strongly suggests that the center of scaling $x = a$ is an isolated blowup point for u. We have thus shown (formally) that the blowup set of u should generically consist of isolated points, in space dimension 2.

If (2.21) applies then we obtain the different asymptotic behavior

$$v(y,s) \sim \frac{\kappa}{2ps}\left(1 - \frac{1}{2}(y\cdot\eta)^2\right). \tag{2.33}$$

In this case v is asymptotically a function of $y \cdot \eta$ only, and it achieves its maximum value on the line $y \cdot \eta = 0$. We believe that when (2.33) holds, the associated solution u of (1.1) blows up on a one-dimensional curve passing through the center of scaling $x = a$, with its tangent line orthogonal to η.

We have shown (formally) that (2.32) and (2.33) are the only possibilities if v tends to zero at an algebraic rate. As in the one dimensional case, however, it should also be possible for v to approach 0 exponentially fast. These non-generic solutions would correspond to trajectories on the stable manifold. It would be interesting to have a classification of their possible asymptotics, since that would give an indication concerning the possible local structure of the blowup set. The simplest case would be to consider solutions with the symmetry $v(y_1, y_2) = v(-y_1, y_2) = v(y_1, -y_2)$, for which the first stable eigenvalue is -1. The associated eigenspace is spanned by

$$e_1^-(y) = h_0 h_4(y_1) \quad , \quad e_2^-(y) = h_0 h_4(y_2),$$
$$e_3^-(y) = h_2(y_1) h_2(y_2).$$

The ansatz

$$v \sim \gamma_1(s) e_1^-(y) + \gamma_2(s) e_2^-(y) + \gamma_3(s) e_3^-(y)$$

leads to a system of the form

$$\begin{aligned} \dot{\gamma_1} &= -\gamma_1 + a\gamma_1^2 + b\gamma_3^2 \\ \dot{\gamma_2} &= -\gamma_2 + a\gamma_2^2 + b\gamma_3^2 \\ \dot{\gamma_3} &= -\gamma_3 + c\gamma_3^2 + 2b\gamma_3(\gamma_1 + \gamma_2), \end{aligned} \tag{2.34}$$

where a, b, and c are certain (explicitly computable) constants. A natural first step would be to classify the possible asymptotic behaviors of solutions of (2.34). Some qualitative results (but not a complete classification) are given in [21].

3 A Program for Rigorous Analysis.

As we have explained, the essential issue is how solutions of

$$v_s - \frac{1}{\rho}\nabla(\rho\nabla v) - v = \frac{p}{2\kappa}v^2 \tag{3.1}$$

behave as they tend to zero (see (1.6) or (2.9)). Center manifold theory is the standard tool for addressing such questions. There are infinite dimensional versions of center manifold theory, see e.g. [6,15,20]. So the question naturally arises whether this theory might apply directly to (3.1).

Unfortunately, it does not. A "standard" center manifold analysis would begin by viewing (3.1) as an O.D.E. in a suitably chosen function space X. The solution would be represented by a version of the variation of parameters integral formula. The space X should be chosen so that (at the very least) the following two conditions apply: a) $\|v(s)\|_X \to 0$ as $s \to \infty$, for the solutions under consideration; and b) $\|v^2\|_X \leq \|v\|_X^2$, so that the nonlinear term v^2 is genuinely quadratic. Condition (a) rules out a translation-invariant space such as $H^l(R^n)$: our solutions $v(y, s)$ converge to 0 as $s \to \infty$ for $| y |\leq C$, but they have

$$lim_{|y|\to\infty} v(y, s) = lim_{|y|\to\infty} w(y, s) - \kappa = -\kappa \tag{3.2}$$

for every $s < \infty$. (We assume for (3.2) that u, the solution of (1.1), satisfies $u(x,t) \to 0$ as $|x| \to \infty$, $0 < t < T$.) Condition (a) would hold for a weighted space such as H^l_ρ (consisting of functions with l derivatives in L^2_ρ). However, it is easy to see that (b) fails for any such space – not only for the exponential weight $\rho(y) = e^{-\frac{|y|^2}{4}}$, but for any weight which decays to zero as $|y| \to \infty$. We doubt the existence of any space satisfying both conditions (a) and (b).

Thus, it seems that center manifold theory does not apply directly to (3.1). In particular, we are not able to prove the existence of a center manifold for (3.1).

We can make progress, however, by focussing more sharply on the specific task at hand. We already *have* a solution of (3.1) which tends to 0 uniformly on compact sets, so existence is not at issue. Moreover, (3.1) is a rather special equation, with a great deal of structure. We shall use the special features of the equation (namely, certain weighted energy estimates), along with ideas borrowed from center manifold theory, to characterize the asymptotic behavior of v.

Our method is illustrated by the following 3×3 O.D.E. example:

$$\dot{z} = z + f_1(x,y,z) \tag{3.3a}$$
$$\dot{x} = -(x+y+z)^2 \tag{3.3b}$$
$$\dot{y} = -y + f_2(x,y,z), \tag{3.3c}$$

where the f_i's, i=1, 2 are assumed to be quadratic in x, y, z. Suppose we know that there exists a trajectory (x,y,z) of (3.3) with the properties:

$$0 < x, y, z \to 0 \quad as \quad t \to \infty. \tag{3.4}$$

Center manifold theory is applicable for this example, and one can use it to get the asymptotic behavior of the (x,y,z) trajectory. However, we can avoid the general theory by using the following elementary O.D.E. Lemma:

Lemma 3.1 *Let $x(t)$, $y(t)$, $z(t)$ be absolutely continuous, real valued functions which are nonnegative and satisfy:*

$$\dot{z} \geq c_0 z - \epsilon(x+y) \tag{3.5a}$$
$$|\dot{x}| \leq \epsilon(x+y+z) \tag{3.5b}$$
$$\dot{y} \leq -c_0 y + \epsilon(x+z), \tag{3.5c}$$

$$x, y, z \to 0 \quad as \quad t \to \infty, \tag{3.5d}$$

where c_0 is any positive constant and ϵ is a sufficiently small positive constant. Then:

either (i) x, y, $z \to 0$ *exponentially fast*
or else, (ii) *there exists a time t_0 such that $z + y \leq b\epsilon x$, for $t \geq t_0$, where b is a positive constant depending only on c_0.*

The proof will be given in the next section.

From the properties (3.4) of the trajectory (x, y, z) and the fact that the f_i's represent quadratic nonlinearities, we have that for any $\epsilon > 0$, there exists a $\delta > 0$ such that for $0 < x,\ y,\ z < \delta$:

$$
\begin{aligned}
|f_i(x, y, z)| &\le \epsilon(x + y + z), \qquad\qquad i = 1, 2,\\
(x + y + z)^2 &< \epsilon(x + y + z).
\end{aligned}
$$

Hence, from (3.3) we obtain:

$$
\begin{aligned}
\dot{z} &\ge (1 - \epsilon)z - \epsilon(x + y)\\
|\ \dot{x}\ | &\le \epsilon(x + y + z)\\
\dot{y} &\le -(1 - \epsilon)y + \epsilon(x + z).
\end{aligned}
\tag{3.6}
$$

Using the O.D.E. Lemma, with say $c_0 = 1/2$, we conclude:

$$z + y < b\epsilon x,$$

and from equation (3.3b) we have that:

$$\dot{x} = -(1 + O(\epsilon))x^2 \qquad \Rightarrow \qquad x = \frac{1}{t} + O(\frac{\epsilon}{t}).$$

Hence to the leading order we get that x decays like $\frac{1}{t}$.

The O.D.E. Lemma plays a similar role in our analysis, although some extra work is needed since the nonlinear term of (3.1) is not quadratic in L^2_ρ (i.e. condition (b) stated above is false in L^2_ρ). To explain more, we now sketch the argument when $n = 1$, $p = 2$. In this case the exact equation for v is:

$$v_s = \mathcal{L}v + v^2. \tag{3.7}$$

Let

π_+ : orthogonal projection onto the unstable subspace of $\mathcal{L}$,
π_0 : orthogonal projection onto the null subspace of $\mathcal{L}$,
π_- : orthogonal projection onto the stable subspace of $\mathcal{L}$,

and $v_+ = \pi_+ v$, $v_0 = \pi_0 v$, $v_- = \pi_- v$ so that $v = v_+ + v_0 + v_-$.

Projecting (3.7) onto $\pi_+ L^2_\rho$ we get:

$$\dot{v}_+ = \mathcal{L}v_+ + \pi_+ v^2.$$

Forming the L^2_ρ inner product with v_+ and using standard inequalities (see Section 4 for details) we arrive at:

$$\frac{d}{dt}\|v_+\| \ge \frac{1}{2}\|v_+\| - \|v^2\|,$$

where $\|\cdot\|$ denotes the L^2_ρ norm. Let $z = \|v_+\|$, $x = \|v_0\|$, $y = \|v_-\|$, $N = \|v^2\|$. Working similarly for v_0, v_- we arrive at a system which is roughly analogous to (3.6):

$$\begin{aligned} \dot{z} &\geq \frac{1}{2}z - N \\ |\,\dot{x}\,| &\leq N \\ \dot{y} &\leq -\frac{1}{2}y + N. \end{aligned} \tag{3.8}$$

We would like to know that $N \leq \epsilon(x+y+z)$, which is equivalent to

$$\int v^4\rho \leq \epsilon^2 \int v^2\rho. \tag{3.9}$$

We could then use the O.D.E. Lemma to conclude that

$$\|v_+\| + \|v_-\| \leq b\epsilon\|v_0\|. \tag{3.10}$$

We have already observed that (3.9) is not true for an arbitrary $v \in L^2_\rho$. The idea of the proof is to use equation (3.7) to show that (3.9) holds for the particular trajectory of interest. Working in this direction we write:

$$\int v^4\rho = \int_{|y|\leq\delta^{-1}} v^4\rho + \int_{|y|\geq\delta^{-1}} v^4\rho, \qquad \text{for } \delta > 0. \tag{3.11}$$

Since $v(y,s) \to 0$ uniformly for $|\,y\,| < C$, as $s \to \infty$ we have that for any ϵ, $\delta > 0$, there is an s_0 such that:

$$\int_{|y|\leq\delta^{-1}} v^4\rho \leq \epsilon^2 \int_{|y|\leq\delta^{-1}} v^2\rho \leq \epsilon^2 \int v^2\rho \qquad \text{for } s \geq s_0.$$

For the second term of (3.11) we write:

$$\int_{|y|\geq\delta^{-1}} v^4\rho \leq \delta^k \int v^4\,|\,y\,|^k\,\rho := \delta^k J^2.$$

To estimate J we multiply (3.7) by $v^3\,|\,y\,|^k\,\rho$ and integrate by parts. After certain calculations (see Section 4 for details) we end up with an inequality of the form:

$$\dot{J} \leq -\theta(k,\delta)J + \epsilon'(\epsilon,\delta)(x+y+z), \tag{3.12}$$

where

$$\begin{aligned} 0 &< \theta(k,\delta) = O(1), \\ 0 &< \epsilon'(\epsilon,\delta) = o(1). \end{aligned}$$

Inequality (3.12) by itself does not contain enough information to allow us to conclude the missing estimate (3.9), but we can couple it with system (3.8). Defining a new variable $\tilde{y} = y + J$, we can now use the O.D.E. Lemma to conclude (3.10). Of course (3.10) is equivalent to (1.13) (see Theorem A stated in the Introduction).

In order to get the O.D.E. for the evolution of the neutral mode, recalling that $v_0(y,s) = \alpha_1(s)h_2(y)$ we project (3.7) onto the neutral subspace of $\mathcal{L}$ (cf (1.14)):

$$\dot{\alpha}_1 = \pi_0(v^2) = \pi_0(v_0^2) + \pi_0(v^2 - v_0^2). \tag{3.13}$$

The main ingredient in the proof of Theorem B is to show that the last term of (3.13) is small compared to the other term. This is done by using (3.10) as well as a new *a priori* estimate for the v equation (Lemma 5.1).

For the proof of Theorem C we repeat the argument of Theorem A not in L^2_ρ but in H^1_ρ. Then, using the Sobolev embedding theorem we conclude that the neutral component of v dominates its large time behavior in the sup norm. Since (3.13) can be solved explicitly in the one dimensional case, we are able to write down the exact profile of $v(y,s)$.

Finally, Theorem D is a consequence of Theorem C and a result of Y. Giga and R. Kohn (see the beginning of Section 7 for more details).

4 Reduction to the Study of the Neutral Modes.

In this section we will give the proof of Theorem A. We recall that we are studying a nonnegative, blowing-up solution of the semilinear heat equation (1.1). We shall assume henceforth that when written in similarity variables (see (1.2)), the solution satisfies

(i) w is nonnegative,

(ii) w is uniformly bounded in space-time, and

(iii) $w(y,s) \to \kappa$ as $s \to \infty$, uniformly on compact sets in y.

From (ii) and (iii) we also have that $v = w - \kappa \to 0$ in L^2_ρ, by the dominated convergence theorem. These conditions are known to be valid for any nonnegative solution of the Cauchy problem (1.1) provided that (a) u is uniformly bounded at infinity (e.g. if $u \to 0$ at infinity); (b) $n \le 2$ or, if $n \ge 3$, $p < \frac{n+2}{n-2}$; and (c) the center of scaling is a blowup point [11-13].

We begin with the proof of the O.D.E. lemma presented in the previous section.

Proof of Lemma 3.1: By rescaling in time, we may assume that $c_0 = 1$. We divide the proof into 5 steps:

step 1 : If it is not true that x, y, $z \to 0$ exponentially fast, there will be a time at which $y < 2\epsilon(x+z)$.

Indeed, if $y \ge 2\epsilon(x+z)$ for all time then from (3.5c) we have

$$\dot{y} \le -y + \epsilon(x+z) \le -y + \frac{y}{2} = -\frac{y}{2}.$$

This implies that $y \to 0$ exponentially fast, and that forces x, z to decay exponentially fast as well.

step 2 : Let $\alpha(t) = y - 2\epsilon(x+z)$. Once the quantity $\alpha(t)$ becomes negative, it will stay nonpositive thereafter.

To prove this, we observe first that (3.5) gives

$$\alpha(t) \ge 0 \quad \Rightarrow \quad \dot{\alpha}(t) \le 0, \tag{4.1}$$

provided ϵ is sufficiently small. Indeed,

$$\dot{\alpha}(t) = \dot{y} - 2\epsilon(\dot{x} + \dot{z}) \leq -(1-4\epsilon^2)y + (\epsilon + 4\epsilon^2)x - (\epsilon - 2\epsilon^2)z.$$

Using the fact that $\alpha(t) \geq 0 \Leftrightarrow -y \leq -2\epsilon(x+z)$, we see that $\alpha(t) \geq 0$ implies

$$\dot{\alpha}(t) \leq -\epsilon(1 - 4\epsilon - 8\epsilon^2)x - \epsilon(3 - 2\epsilon - 8\epsilon^2)z \leq 0$$

for sufficiently small ϵ, and assertion (4.1) has been proved.

Let $\alpha_+(t)$ be the positive part of $\alpha(t)$, i.e.

$$\alpha_+(t) = \begin{cases} \alpha(t) & \text{if } \alpha(t) \geq 0 \\ 0 & \text{if } \alpha(t) \leq 0. \end{cases}$$

It is a standard fact that $\alpha_+(t)$ is an absolutely continous function. Moreover, we have

$$\dot{\alpha}_+(t) = \begin{cases} \dot{\alpha}(t) & \text{if } \alpha(t) \geq 0 \\ 0 & \text{if } \alpha(t) \leq 0 \end{cases}$$

almost everywhere, see e.g [14]. Hence we always have that $\dot{\alpha}_+(t) \leq 0$. Now suppose that t' is a time at which $\alpha(t') < 0$. Using the fundamental theorem of calculus we compute for $t \geq t'$:

$$\alpha_+(t) = \alpha_+(t) - \alpha_+(t') = \int_{t'}^{t} \dot{\alpha}_+(s)ds \leq 0.$$

This implies that the positive part of $\alpha(t)$ is necessarily 0 for $t \geq t'$, i.e. $\alpha(t)$ is nonpositive for $t \geq t'$. We have thus shown that

$$y \leq 2\epsilon(x+z) \quad \text{for } t \text{ sufficiently large.} \tag{4.2}$$

<u>*step 3*</u> : There exists some time at which $z < 2\epsilon(x+y)$.

If not, then (3.5a) forces z to grow exponentially fast, contradicting (3.5d).

<u>*step 4*</u> : Let $\beta(t) = z - 2\epsilon(x+y)$. Once the quantity $\beta(t)$ becomes negative, it will stay nonpositive thereafter.

If not, let t^* be some time when $\beta > 0$. Then, so long as $\beta(t) \geq 0$ we compute (as in step 2):

$$\dot{\beta}(t) = \dot{z} - 2\epsilon(\dot{x} + \dot{y}) \geq (1-4\epsilon^2)z - (\epsilon + 4\epsilon^2)x + (\epsilon - 2\epsilon^2)y \geq$$

$$\geq \epsilon(1 - 4\epsilon - 8\epsilon^2)x + \epsilon(3 - 2\epsilon - 8\epsilon^2)y \geq 0.$$

But then, for $t \geq t^*$ we have that $\beta(t)$ is a positive quantity with a nonnegative slope, contradicting the fact that $\beta(t) \to 0$ as $t \to \infty$. We conclude that

$$z \leq 2\epsilon(x+y), \quad \text{for } t \text{ sufficiently large.} \tag{4.3}$$

<u>*step 5*</u> : Putting together (4.2) and (4.3), and taking ε sufficiently small, we get the desired result.

Examination of the proof shows that we can choose the constant b to be equal to 10 (or $\frac{10}{c_0}$ for $c_0 \neq 1$); its exact value is not needed in the proofs to follow.

□

In order to linearize the w-equation (1.3) about $w = \kappa$, we set $v = w - \kappa$ and consider the Taylor expansion of the nonlinear term of (1.3). Since $v + \kappa \geq 0$ we have

$$(v+\kappa)^p = \kappa^p + p\kappa^{p-1}v + \frac{1}{2}p(p-1)(\kappa+\phi)^{p-2}v^2,$$

where ϕ is between 0 and v. Let

$$c(\phi,p) := \frac{1}{2}p(p-1)(\kappa+\phi)^{p-2}.$$

We need a bound on $c(\phi,p)$. Using the fact that $\mid v \mid \leq M$ we show

Lemma 4.1 *For any $p > 1$*

$$0 \leq c(\phi,p) \leq C,$$

where C is a constant depending only on M and p.

Proof: Clearly $0 \leq c(\phi,p)$. To obtain the upper bound we distinguish 3 cases.
case (i): $p \geq 2$. Since $\mid v \mid < M$ we also have that $\mid \phi \mid < M$; hence $c(\phi,p) < c(M,p)$.
case (ii): $1 < p < 2$ and $\mid \phi \mid \leq \frac{\kappa}{2}$. Then

$$\phi + \kappa \geq \frac{\kappa}{2} \quad \Rightarrow \quad c(\phi,p) \leq \frac{1}{2}p(p-1)\left(\frac{\kappa}{2}\right)^{p-2}.$$

case (iii): $1 < p < 2$ and $\mid \phi \mid > \frac{\kappa}{2}$. We have

$$c(\phi,p)v^2 = (v+\kappa)^p - \kappa^p - p\kappa^{p-1}v.$$

But $\mid v \mid \geq \mid \phi \mid > \frac{\kappa}{2}$, and therefore

$$(v+\kappa)^p < 3^p \mid v \mid^p, \qquad p\kappa^{p-1} \mid v \mid < p2^{p-1} \mid v \mid^p.$$

It follows that

$$c(\phi,p)v^2 < (3^p + p2^{p-1}) \mid v \mid^p,$$

or

$$c(\phi,p) < (3^p + p2^{p-1}) \mid v \mid^{p-2} < (3^p + p2^{p-1}) \left(\frac{\kappa}{2}\right)^{p-2}.$$

Hence in all cases we have $c(\phi,p) \leq C$ with C depending on M and p.

□

We also note that

$$c(0,p) = \frac{p}{2\kappa}. \tag{4.4}$$

Using the Taylor expansion we can rewrite (1.3) as

$$v_s = \frac{1}{\rho}\nabla(\rho\nabla v) + v + c(\phi,p)v^2,$$

or equivalently

$$\dot{v} = \mathcal{L}v + c(\phi,p)v^2, \tag{4.5}$$

where $\mathcal{L}v = \frac{1}{\rho}\nabla(\rho\nabla v) + v$ and "$\cdot$" denotes the time derivative.

We are now ready to give the proof of Theorem A, following the ideas sketched in the previous section. The only things we will use in the proof are the v-equation (4.5) and the knowledge that (i) $v(y,s) \geq -\kappa$ for all $(y,s) \in R^n \times (0,\infty)$, (ii) $| v(y,s) |< M$ for all $(y,s) \in R^n \times (0,\infty)$, and (iii) v exists for all time and $v(y,s) \to 0$ as $s \to \infty$, uniformly on $| y |< C$. These of course follow immediately from the corresponding hypotheses on w stated at the beginning of the section.

Proof of Theorem A : Let v_+, v_0, v_- represent the ustable, neutral and stable part of v as defined in Section 3. Projecting (4.5) onto the unstable subspace of $\mathcal{L}$ we get

$$\dot{v}_+ = \mathcal{L}v_+ + \pi_+\left[c(\phi,p)v^2\right]. \tag{4.6}$$

Next, we multiply (4.6) by $v_+\rho$ and integrate over R^n to obtain

$$\frac{1}{2}\frac{d}{ds}\int v_+^2\rho = \int \mathcal{L}v_+ \cdot v_+\rho + \int \pi_+\left[c(\phi,p)v^2\right]v_+\rho. \tag{4.7}$$

For the first term of the right hand side we observe that

$$\int \mathcal{L}v_+ \cdot v_+\rho \;\geq\; \frac{1}{2}\int v_+^2\rho,$$

since $\frac{1}{2}$ is the smallest positive eigenvalue of $\mathcal{L}$. We can estimate the last term of the right hand side of (4.7) by

$$\left| \int \pi_+\left(c(\phi,p)v^2\right)v_+\rho \right| \leq \left(\int\left(\pi_+\left[c(\phi,p)v^2\right]\right)^2\rho\right)^{\frac{1}{2}}\left(\int v_+^2\rho\right)^{\frac{1}{2}} \leq \left(\int c^2(\phi,p)v^4\rho\right)^{\frac{1}{2}}\left(\int v_+^2\rho\right)^{\frac{1}{2}}.$$

Hence we conclude that

$$\frac{1}{2}\frac{d}{ds}\int v_+^2 \geq \frac{1}{2}\int v_+^2\rho - \left(\int c^2(\phi,p)v^4\rho\right)^{\frac{1}{2}}\left(\int v_+^2\rho\right)^{\frac{1}{2}}.$$

If we set

$$z = \left(\int v_+^2\rho\right)^{\frac{1}{2}}, \qquad N = \left(\int v^4\rho\right)^{\frac{1}{2}},$$

then the above inequality can be written as:

$$\frac{1}{2}(\dot{z^2}) \geq \frac{1}{2}z^2 - C\dot{z}N.$$

where we also used the fact that $c(\phi, p) < C$. After simplifying we get

$$\dot{z} \geq \frac{1}{2}z - CN. \tag{4.8}$$

(Notice that on any set where $z = 0$, $\dot{z} = 0$ a.e.; thus (4.8) is an almost everywhere valid relation between two measurable functions.) Working similarly for the neutral component (v_0) and the stable component (v_-) of v we end up with the system:

$$\begin{aligned} \dot{z} &\geq \frac{1}{2}z - CN \\ |\, \dot{x} \,| &\leq CN \\ \dot{y} &\leq -\frac{1}{2}y + CN, \end{aligned} \tag{4.9}$$

where $x = (\int v_0^2 \rho)^{\frac{1}{2}}$ and $y = (\int v_-^2 \rho)^{\frac{1}{2}}$. (The auxiliary function y just introduced is not to be confused with the space variable y of the PDE.)

As explained in the previous section, if we knew that $N \leq \epsilon(x+y+z)$ we could appeal to the O.D.E. lemma to conclude the desired estimate. Since we do not have such an inequality we estimate N as follows. Given any $\epsilon > 0$ and any $\delta > 0$ (both will chosen small in the sequel) there is a time s_0 such that

$$N^2 = \int v^4 \rho = \int_{|y| \leq \delta^{-1}} v^4 \rho + \int_{|y| \geq \delta^{-1}} v^4 \rho \leq \epsilon^2 \int v^2 \rho + \delta^k \int v^4 \,|\, y \,|^k \rho, \quad for \quad s \geq s_0.$$

We use here the fact that $v \to 0$ as $s \to \infty$ uniformly on the compact set $|\, y \,| \leq \delta^{-1}$. The exponent k which appears in the last term is a positive integer, otherwise arbitrary (later on we will impose certain restrictions on k). Let

$$J := \left(\int v^4 \,|\, y \,|^k \rho \right)^{\frac{1}{2}},$$

so that the above estimate can be rewritten as

$$N \leq \epsilon(x + y + z) + \delta^{\frac{k}{2}} J \qquad for \quad s \geq s_0. \tag{4.10}$$

In order to find an estimate for J we multiply equation (4.5) by $v^3 \,|\, y \,|^k \rho$ and then integrate over R^n to get

$$\frac{1}{4}\frac{d}{ds}\int v^4 \,|\, y \,|^k \rho = \int \nabla(\rho \nabla v) v^3 \,|\, y \,|^k + \int v^4 \,|\, y \,|^k \rho + \int c(\phi, p) v^5 \,|\, y \,|^k \rho. \tag{4.11}$$

For the first term of the right hand side we integrate by parts to get, after some calculation,

$$\int \nabla(\rho \nabla v) v^3 \,|\, y \,|^k = -3 \int v^2 \,|\, \nabla v \,|^2 |\, y \,|^k \rho + \frac{k}{4}(k - 2 + n) \int v^4 \,|\, y \,|^{k-2} \rho - \frac{k}{8} \int v^4 \,|\, y \,|^k \rho.$$

For the last term of the right hand side of (4.11), using the fact that $c(\phi,p)v \leq CM$ we get

$$\int c(\phi,p)v^5 \mid y \mid^k \rho \leq CM \int v^4 \mid y \mid^k \rho.$$

Hence, from (4.11) we conclude

$$\frac{1}{4}\frac{d}{ds}\int v^4 \mid y \mid^k \rho \leq -3\int v^2 \mid \nabla v \mid^2 \mid y \mid^k \rho +$$

$$+\frac{k}{4}(k-2+n)\int v^4 \mid y \mid^{k-2} \rho - (\frac{k}{8}-1-CM)\int v^4 \mid y \mid^k \rho. \tag{4.12}$$

Using Schwartz's inequality we have that

$$\int v^4 \mid y \mid^{k-2} \rho \leq \left(\int v^4 \mid y \mid^k \rho\right)^{\frac{1}{2}} \left(\int v^4 \mid y \mid^{k-4} \rho\right)^{\frac{1}{2}}. \tag{4.13}$$

Omitting the first term of the right hand side of (4.12), which is negative definite, using (4.13) and recalling that $J = \left(\int v^4 \mid y \mid^k \rho\right)^{\frac{1}{2}}$ we obtain

$$\frac{1}{2}\dot{J} \leq -(\frac{k}{8}-1-CM)J + \frac{k}{4}(k-2+n)\left(\int v^4 \mid y \mid^{k-4} \rho\right)^{\frac{1}{2}}. \tag{4.14}$$

Next we estimate the last term of (4.14):

$$\left(\int v^4 \mid y \mid^{k-4} \rho\right)^{\frac{1}{2}} \leq \left(\int_{|y|\leq\delta^{-1}} v^4 \mid y \mid^{k-4} \rho\right)^{\frac{1}{2}} + \left(\int_{|y|\geq\delta^{-1}} v^4 \mid y \mid^{k-4} \rho\right)^{\frac{1}{2}}$$

$$\leq \delta^{2-\frac{k}{2}}\left(\int_{|y|\leq\delta^{-1}} v^4 \rho\right)^{\frac{1}{2}} + \delta^2\left(\int_{|y|\geq\delta^{-1}} v^4 \mid y \mid^k \rho\right)^{\frac{1}{2}}$$

$$\leq \epsilon\delta^{2-\frac{k}{2}}\left(\int_{|y|\leq\delta^{-1}} v^2 \rho\right)^{\frac{1}{2}} + \delta^2\left(\int v^4 \mid y \mid^k \rho\right)^{\frac{1}{2}}$$

$$\leq \epsilon\delta^{2-\frac{k}{2}}(x+y+z) + \delta^2 J, \quad for \quad s \geq s_0,$$

where s_0 is the same as in (4.10). Using the above estimate and (4.14) we get

$$\dot{J} \leq -\theta J + \epsilon'(x+y+z), \tag{4.15}$$

with

$$\theta = \theta(k,\delta,n) = \frac{k}{4} - 2 - 2CM - \frac{k\delta^2}{2}(k+n-2), \tag{4.16}$$

$$\epsilon' = \epsilon'(\epsilon, \delta, k, n) = \frac{1}{2}\epsilon\delta^{2-\frac{k}{2}}k(k+n-2). \tag{4.17}$$

Up to this point, we have made no assumptions about the choices of k, δ, and ϵ. We now put certain restrictions on them in order to finish the proof. By first choosing k large, then choosing δ small, we can always make $\theta \geq \frac{1}{2}$. Hence from (4.15) we obtain

$$\dot{J} \leq -\frac{1}{2}J + \epsilon'(x+y+z). \tag{4.18}$$

Putting together (4.9), (4.10) and (4.18) we get the system

$$\begin{aligned} \dot{z} &\geq \frac{1}{2}z - \epsilon C(x+y+z) - \delta^{\frac{k}{2}}CJ \\ |\dot{x}| &\leq \epsilon C(x+y+z) + \delta^{\frac{k}{2}}CJ \\ \dot{y} &\leq -\frac{1}{2}y + \epsilon C(x+y+z) + \delta^{\frac{k}{2}}CJ \\ \dot{J} &\leq -\frac{1}{2}J + \epsilon'(x+y+z). \end{aligned} \tag{4.19}$$

Let

$$\tilde{y} = y + J, \qquad \hat{\epsilon} = max\left\{\epsilon C + \epsilon', \quad \delta^{\frac{k}{2}}C\right\}.$$

We can make $\hat{\epsilon}$ arbitrarily small by taking ϵ and δ small enough. After adding the last two inequalities of system (4.19) we can rewrite it as

$$\begin{aligned} \dot{z} &\geq (\frac{1}{2} - \hat{\epsilon})z - \hat{\epsilon}(x+\tilde{y}) \\ |\dot{x}| &\leq \hat{\epsilon}(x+\tilde{y}+z) \\ \dot{\tilde{y}} &\leq -(\frac{1}{2} - \hat{\epsilon})\tilde{y} + \hat{\epsilon}(x+z). \end{aligned} \tag{4.20}$$

Using the O.D.E. lemma, we conclude that

$$z + \tilde{y} \leq b\hat{\epsilon}x.$$

In particular we have that

$$\|v_+\|_{L^2_\rho} + \|v_-\|_{L^2_\rho} \leq b\hat{\epsilon}\|v_0\|_{L^2_\rho}, \tag{4.21}$$

and the conclusion of Theorem A follows.

To avoid any possible confusion, we recapitulate the argument. Suppose we are given an $\epsilon_0 > 0$. We want to show that there exists a time s_0 such that:

$$\|v_+\|_{L^2_\rho} + \|v_-\|_{L^2_\rho} \leq \epsilon_0\|v_0\|_{L^2_\rho} \qquad for \quad s \geq s_0. \tag{4.22}$$

- We first choose k, from (4.16), such that $\frac{k}{4} - 2 - 2CM \geq 1$.
- Then, we choose a δ small enough so that

(i) $\frac{k\delta^2}{2}(k+n-2) \leq \frac{1}{2}$,

(ii) $C\delta^{\frac{k}{2}} < \epsilon_0/b$.
From requirement (i) we have that $\theta \geq 1/2$.
• Next, from (4.17) we choose an ϵ such that $C\epsilon + \epsilon' < \epsilon_0/b$.
For these choices of ϵ, δ, k there exists a time s_0, after which all the above calculations are valid, in particular (4.21) is true. Moreover the way we chose them guarantees that $\hat{\epsilon} < \epsilon_0/b$, therefore (4.22) follows from (4.21).

□

5 Evolution of the Neutral Modes.

Theorem A asserts that if v does not tend to zero exponentially fast then the dominant modes are the neutral ones. Our formal analysis in section 2 was based on a study of the O.D.E. which is asymptotically satisfied by these neutral modes. The goal of this section is to prove Theorem B, which asserts the asymptotic validity of this O.D.E.

We shall assume throughout this section that v does not decay exponentially fast. For such v we prove the following

Lemma 5.1 *There exist $\delta_0 > 0$ and an integer $k > 4$ with the following property: given any $0 < \delta < \delta_0$, there exists a time s^* such that*

$$\int v^2 \mid y \mid^k \rho \leq c_0(k)\delta^{4-k} \int v_0^2 \rho, \qquad for \quad s \geq s^*, \tag{5.1}$$

where $c_0(k)$ is a positive constant depending only on k.

Proof: We begin by deriving a differential inequality for the left hand side of (5.1). Recall that v solves

$$\dot{v} = \frac{1}{\rho}\nabla(\rho\nabla v) + v + c(\phi,p)v^2. \tag{5.2}$$

We multiply (5.2) by $v \mid y \mid^k \rho$, where k is any positive integer, and integrate over all R^n to get

$$\frac{1}{2}\frac{d}{ds}\int v^2 \mid y \mid^k \rho = \int \nabla(\rho\nabla v)v \mid y \mid^k + \int v^2 \mid y \mid^k \rho + \int c(\phi,p)v^3 \mid y \mid^k \rho. \tag{5.3}$$

For the first term on the right hand side of (5.3) some calculation gives

$$\int \nabla(\rho\nabla v)v \mid y \mid^k = \frac{k}{2}(k+n-2)\int \mid y \mid^{k-2} v^2\rho - \frac{k}{4}\int \mid y \mid^k v^2\rho.$$

For the last term of the right hand side of (5.3) we use the fact that $c(\phi,p)v < CM$. We thus get

$$\frac{1}{2}\frac{d}{ds}\int v^2 \mid y \mid^k \rho \leq -(\frac{k}{4} - 1 - CM)\int v^2 \mid y \mid^k \rho + \frac{k}{2}(k+n-2)\int v^2 \mid y \mid^{k-2} \rho. \tag{5.4}$$

We further estimate the last term of (5.4) by using Schwartz's inequality:

$$\mid \int v^2 \mid y \mid^{k-2} \rho \mid \leq \left(\int v^2 \mid y \mid^k \rho\right)^{1/2} \left(\int v^2 \mid y \mid^{k-4} \rho\right)^{1/2}.$$

Let $I = \left(\int v^2 \mid y \mid^k \rho\right)^{1/2}$. Then from (5.4) we obtain

$$\dot{I} \leq -(\frac{k}{4} - 1 - CM)I + \frac{k}{2}(k+n-2)\left(\int v^2 \mid y \mid^{k-4} \rho\right)^{1/2}. \tag{5.5}$$

For the last term of (5.5), and for any $\delta > 0$ and $k > 4$, we have

$$\left(\int v^2 \mid y \mid^{k-4} \rho\right)^{1/2} \leq \left(\int_{|y|\leq\delta^{-1}} v^2 \mid y \mid^{k-4} \rho\right)^{1/2} + \left(\int_{|y|\geq\delta^{-1}} v^2 \mid y \mid^{k-4} \rho\right)^{1/2}$$

$$\leq \delta^{2-\frac{k}{2}} \left(\int v^2 \rho\right)^{\frac{1}{2}} + \delta^2 I \leq \delta^{2-\frac{k}{2}}(x+y+z) + \delta^2 I \leq 2\delta^{2-\frac{k}{2}} x + \delta^2 I.$$

Here we use the notation of the previous sections, namely that

$$\int v^2 \rho = x^2 + y^2 + z^2.$$

In the last step in the above inequalities, we used (4.22) with $\epsilon_0 = 1$. So from (5.5) we get

$$\dot{I} \leq -\theta I + d\delta^{2-\frac{k}{2}} x, \tag{5.6}$$

with

$$\theta = \frac{k}{4} - 1 - CM - \frac{k}{2}(k+n-2)\delta^2, \tag{5.7}$$

and

$$d = k(k+n-2). \tag{5.8}$$

From (5.7) it is evident that there exist an integer $k > 4$ and a positive number δ_0 with the property that $0 < \delta < \delta_0$ implies $\theta \geq 1$. Hence, for these choices, we have from (5.6)

$$\dot{I} \leq -I + d\delta^{2-\frac{k}{2}} x. \tag{5.9}$$

The next step is to couple (5.9) with the differential inequality controlling the evolution of x. From (4.9) we have

$$\mid \dot{x} \mid \leq CN. \tag{5.10}$$

For N, using the facts that $\mid v \mid \leq M$ and that $v \to 0$ uniformly on compact sets, we write

$$N^2 = \int v^4 \rho = \int_{|y|\leq\delta^{-1}} v^4 \rho + \int_{|y|\geq\delta^{-1}} v^4 \rho \leq \epsilon^2 \int v^2 \rho + M^2 \int_{|y|\geq\delta^{-1}} v^2 \rho$$

$$\leq \epsilon^2 \int v^2 \rho + M^2 \delta^k \int v^2 \mid y \mid^k \rho.$$

Thus we have

$$N \leq M\delta^{\frac{k}{2}} I + \epsilon(x+y+z).$$

After using (4.22) with $\epsilon_0 = 1$ we end up with

$$N \leq M\delta^{\frac{k}{2}} I + 2\epsilon x. \tag{5.11}$$

Putting together (5.9),(5.10) and (5.11) we arrive at the system:

$$\dot{I} \leq -I + d\delta^{2-\frac{k}{2}} x \tag{5.12a}$$

$$|\, \dot{x} \,| \leq CM\delta^{\frac{k}{2}} I + 2C\epsilon x. \tag{5.12b}$$

To conclude the proof we argue as as in step 2 of the O.D.E. Lemma in section 4. We observe that there is a time s^* at which $I < 2d\delta^{2-\frac{k}{2}} x$. (If not, then from (5.12a) I would tend to zero exponentially fast, and that would force x to decay exponentially fast as well.) We assert that once the quantity $\gamma(s) = I - 2d\delta^{2-\frac{k}{2}} x$ becomes negative it will stay nonpositive thereafter. To show this, it is sufficient to prove that

$$\gamma(s) \geq 0 \quad \Rightarrow \quad \dot{\gamma}(s) \leq 0. \tag{5.13}$$

Indeed, if $\gamma(s) \geq 0$ then arguing as in step 2 of the O.D.E. lemma we end up after some calculation with

$$\dot{\gamma}(s) = \dot{I} - 2d\delta^{2-\frac{k}{2}} \dot{x} \leq -d\delta^{2-\frac{k}{2}} (1 - 4dCM\delta^2 - 4C\epsilon) x \leq 0,$$

for suitably small δ, ϵ. This proves (5.13). We conclude that γ is eventually nonpositive, i.e.

$$I \leq 2d\delta^{2-\frac{k}{2}} x \qquad for \quad s \geq s^*. \tag{5.14}$$

Clearly, (5.1) follows from (5.14) with

$$c_0(k) = 4d^2 = 4k^2(k+n-2)^2.$$

□

We are now ready to justify the O.D.E. satisfied by the neutral modes.

Proof of Theorem B: Projecting equation (4.5) onto e_j^0 we get

$$\dot{\alpha}_j(s) = \pi_j^0 \left(c(\phi, p) v^2 \right) = \pi_j^0 \left(c(0, p) v_0^2 \right) + E = \frac{p}{2\kappa} \pi_j^0 v_0^2 + E, \tag{5.15}$$

where we used (4.4), and

$$E := \pi_j^0 \left(c(\phi, p) v^2 - c(0, p) v_0^2 \right).$$

Thus the theorem will be proved once we have shown that

$$|\, E \,| \leq \epsilon \left(\int v_0^2 \rho \right) = \epsilon \left(\sum_{j=1}^{m} \alpha_j^2 \right). \tag{5.16}$$

From Lemma 2.2, we see that the e_j^0's are either of the form

$$e_j^0(y) = C_1(\frac{1}{2}y_i^2 - 1), \qquad i = 1, 2, \ldots, n,$$

or else of the form

$$e_j^0(y) = C_2 y_l y_k, \qquad l \neq k, \quad l, k = 1, 2, \ldots, n,$$

where C_1, C_2 are suitable normalizing constants. In the first case we have

$$\begin{aligned} | E | &= | \int \left(c(\phi,p)v^2 - c(0,p)v_0^2 \right) e_j^0(y)\rho | \\ &\leq C_1 | \int \left(c(\phi,p)v^2 - c(0,p)v_0^2 \right) \rho | + \frac{C_1}{2} \int | c(\phi,p)v^2 - c(0,p)v_0^2 | \, | y |^2 \rho . \end{aligned}$$

In the second case we have

$$| E | = | \int \left(c(\phi,p)v^2 - c(0,p)v_0^2 \right) e_j^0(y)\rho | \leq C_2 \int | c(\phi,p)v^2 - c(0,p)v_0^2 | \, | y |^2 \rho.$$

Hence, in all cases it is enough to prove

$$| \int \left(c(\phi,p)v^2 - c(0,p)v_0^2 \right) \rho | \leq \epsilon \int v_0^2 \rho \tag{5.17}$$

$$\int | c(\phi,p)v^2 - c(0,p)v_0^2 | \, | y |^2 \rho \leq \epsilon \int v_0^2 \rho. \tag{5.18}$$

Proof of (5.17): We have

$$| \int \left(c(\phi,p)v^2 - c(0,p)v_0^2 \right) \rho | \leq \int | c(\phi,p) - c(0,p) | \, v^2 \rho + c(0,p) | \int (v^2 - v_0^2)\rho | . \tag{5.19}$$

For the second term of the right hand side of (5.19) we have

$$c(0,p) | \int (v^2 - v_0^2)\rho | = \frac{p}{2\kappa} \left(\int v_+^2 \rho + \int v_-^2 \rho \right) \leq \epsilon' \frac{p}{2\kappa} \int v_0^2 \rho, \tag{5.20}$$

where in the last step we used (4.22). For the first term of the right hand side of (5.19) we write

$$\begin{aligned} \int | c(\phi,p) - c(0,p) | \, v^2 \rho &= \int_{|y| \leq \delta^{-1}} | c(\phi,p) - c(0,p) | \, v^2 \rho \\ &\quad + \int_{|y| \geq \delta^{-1}} | c(\phi,p) - c(0,p) | \, v^2 \rho. \end{aligned} \tag{5.21}$$

Since $c(\phi,p) = \frac{1}{2}p(p-1)(\kappa + \phi)^{p-2}$ with ϕ between 0 and v, and $v \to 0$ uniformly on $| y | \leq \delta^{-1}$, we have that

$$sup_{|y| \leq \delta^{-1}} | c(\phi,p) - c(0,p) | \leq \epsilon' \quad \text{for } s \geq s',$$

for sufficiently large s'. Hence,

$$\int_{|y|\le\delta^{-1}} | c(\phi,p)-c(0,p) | v^2\rho \le \epsilon' \int v^2\rho \le 2\epsilon' \int v_0^2\rho, \tag{5.22}$$

where once more we used (4.22). For the second term of (5.21) we write

$$\int_{|y|\ge\delta^{-1}} | c(\phi,p)-c(0,p) | v^2\rho \le 2C \int_{|y|\ge\delta^{-1}} v^2\rho$$

$$\le 2C\delta^k \int v^2 | y |^k \rho \le 2Cc_0(k)\delta^4 \int v_0^2\rho, \tag{5.23}$$

where we used the fact that $c(\phi,p) < C$ as well as Lemma 5.1.

From (5.19)-(5.23) we conclude

$$| \int \left(c(\phi,p)v^2 - c(0,p)v_0^2\right) \rho |\le \left(\epsilon' \frac{p}{2\kappa} + 2\epsilon' + 2Cc_0(k)\delta^4\right) \int v_0^2\rho.$$

Since ϵ' and δ can be chosen arbitrarily small, (5.17) has been proved.

Proof of (5.18): We have

$$\int | c(\phi,p)v^2 - c(0,p)v_0^2 | | y |^2 \rho \le \int | c(\phi,p)-c(0,p) | v^2 | y |^2 \rho$$

$$+ c(0,p) \int | v^2 - v_0^2 | | y |^2 \rho \ := I_1 + c(0,p)I_2. \tag{5.24}$$

We now estimate each of the above terms separately. For I_1 we have

$$I_1 = \int_{|y|\le\delta^{-1}} | c(\phi,p)-c(0,p) | v^2 | y |^2 \rho$$

$$+ \int_{|y|\ge\delta^{-1}} | c(\phi,p)-c(0,p) | v^2 | y |^2 \rho := I_{11} + I_{12}. \tag{5.25}$$

For I_{11} we estimate

$$I_{11} \le \delta^{-2} \int_{|y|\le\delta^{-1}} | c(\phi,p)-c(0,p) | v^2\rho \le 2\epsilon'\delta^{-2} \int v_0^2\rho, \tag{5.26}$$

where we used (5.22). For I_{12} we obtain

$$I_{12} \le 2C\delta^{k-2} \int v^2 | y |^k \rho \le 2Cc_0(k)\delta^2 \int v_0^2\rho, \tag{5.27}$$

by using Lemma 5.1. Hence, from (5.26), (5.27) we have

$$I_1 \le \left(2\epsilon'\delta^{-2} + 2Cc_0(k)\delta^2\right) \int v_0^2\rho. \tag{5.28}$$

We estimate I_2 similarly:

$$I_2 \le \int_{|y|\le\delta^{-1}} | v^2 - v_0^2 || y |^2 \rho + \int_{|y|\ge\delta^{-1}} | v^2 - v_0^2 || y |^2 \rho \ := I_{21} + I_{22}. \tag{5.29}$$

To estimate I_{21} we observe that since $v = v_+ + v_0 + v_-$, we have

$$v^2 - v_0^2 = (v_+ + v_-)^2 + 2v_0(v_+ + v_-).$$

Hence, we can write

$$I_{21} \leq \int_{|y|\leq\delta^{-1}} (v_+ + v_-)^2 \mid y \mid^2 \rho + 2\int_{|y\leq\delta^{-1}} \mid v_0(v_+ + v_-) \mid \mid y \mid^2 \rho$$

$$\leq \delta^{-2} \int (v_+ + v_-)^2\rho + 2\left(\int v_0^2 \mid y \mid^4 \rho\right)^{1/2} \left(\int (v_+ + v_-)^2\rho\right)^{1/2}.$$

Using (4.22) we get

$$\int (v_+ + v_-)^2\rho \leq \epsilon' \int v_0^2\rho.$$

Next, we note that since the neutral subspace of $\mathcal{L}$ is finite dimensional, all norms on it are equivalent; therefore

$$\int v_0^2 \mid y \mid^4 \rho \leq \hat{C}^2 \int v_0^2\rho, \tag{5.30}$$

for some constant $\hat{C}$. We conclude that

$$I_{21} \leq (\epsilon'\delta^{-2} + 2\epsilon'^{1/2}\hat{C}) \int v_0^2\rho. \tag{5.31}$$

We finally estimate I_{22}:

$$I_{22} \leq \int_{|y|\geq\delta^{-1}} v^2 \mid y \mid^2 \rho + \int_{|y|\geq\delta^{-1}} v_0^2 \mid y \mid^2 \rho \leq \delta^{k-2} \int v^2 \mid y \mid^k \rho + \delta^{k-2} \int v_0^2 \mid y \mid^k \rho$$

$$\leq c_0(k)\delta^2 \int v_0^2\rho + \delta^{k-2}\hat{C}(k) \int v_0^2\rho, \tag{5.32}$$

where we used Lemma 5.1 and the equivalence of the norms for v_0. From (5.29), (5.31), (5.32) we get:

$$I_2 \leq \left(\epsilon'\delta^{-2} + 2\epsilon'^{1/2}\hat{C} + c_0(k)\delta^2 + \delta^{k-2}\hat{C}(k)\right) \int v_0^2\rho \tag{5.33}$$

Putting together (5.24), (5.28) and (5.33) with ϵ' and δ chosen sufficiently small, we easily complete the proof of (5.18). Notice that one has to choose δ first and then ϵ', so that terms like $\epsilon'\delta^{-2}$ are also small. This is always possible, since ϵ' and δ are independent of each other. The value of k is fixed, being determined by Lemma 5.1.

□

In the special case $n = 1$, (5.15) reads:

$$\dot{\alpha}_1(s) = \frac{2c_2p}{\kappa}\alpha_1^2(s) + O(\epsilon\alpha_1^2), \qquad s \geq s_1, \tag{5.34}$$

with $c_2 = \frac{1}{2}\pi^{-1/4}$. Notice that except for the error term, this is the same as (2.12) which we derived formally in section 2. The solution of (5.34) satisfies

$$\alpha_1 = \left(-[\frac{2c_2 p}{\kappa} + O(\epsilon)](s - s_1) + \alpha_1^{-1}(s_1)\right)^{-1}.$$

Since $\alpha_1(s)$ exists for all time we conclude that $\alpha_1(s_1) < 0$, and that as $s \to \infty$

$$\alpha_1(s) = -\frac{\kappa}{2c_2 ps} + O(\frac{\epsilon}{s}).$$

Recalling that $v_0(y,s) = \alpha_1(s)h_2(y)$, with $h_2(y) = c_2(\frac{1}{2}y^2 - 1)$ (see Section 2), we have proved

Proposition 5.1 *Consider the case of one space dimension, and assume that v does not decay exponentially fast. Then given any ϵ there exists an s_1 after which*

$$v_0(y,s) = \left(\frac{\kappa}{2ps} + O(\frac{\epsilon}{s})\right)(1 - \frac{1}{2}y^2). \tag{5.35}$$

In space dimension $n = 2$, (5.15) reads:

$$\begin{aligned}
\dot{\alpha}_1 &= \frac{p}{2\kappa}(4c_2c_0\alpha_1^2 + 2c_2c_0\alpha_3^2) + O(\epsilon\alpha_1^2 + \epsilon\alpha_2^2 + \epsilon\alpha_3^2) \\
\dot{\alpha}_2 &= \frac{p}{2\kappa}(4c_2c_0\alpha_2^2 + 2c_2c_0\alpha_3^2) + O(\epsilon\alpha_1^2 + \epsilon\alpha_2^2 + \epsilon\alpha_3^2) \\
\dot{\alpha}_3 &= \frac{p}{2\kappa}(4c_2c_0\alpha_1\alpha_3 + 4c_2c_0\alpha_2\alpha_3) + O(\epsilon\alpha_1^2 + \epsilon\alpha_2^2 + \epsilon\alpha_3^2)
\end{aligned} \tag{5.36}$$

Again, (5.36) is to leading order identical to (2.19). A result analogous to Proposition 5.1 for the two-dimensional case will be presented in [7].

6 Center Manifold Analysis in H_ρ^1.

We have already established the asymptotic behavior of $v(y,s)$ in L_ρ^2. It is natural to ask also for a result that is uniform in y, at least on compact sets. For this one must work in a higher Sobolev space: specifically, in R^n one must work in H_ρ^l with $l > n/2$. We discuss here only $n = 1$ for which H_ρ^1 suffices. Our eventual goal is the proof of Theorem C.

Let us first recall the equation for v:

$$\dot{v} = \mathcal{L}v + c(\phi, p)v^2, \tag{6.1}$$

or equivalently

$$\dot{v} = \frac{1}{\rho}\frac{d}{dy}\left(\rho\frac{d}{dy}v\right) - \beta(v + \kappa) + (v + \kappa)^p, \tag{6.2}$$

where $\beta = \frac{1}{p-1}$, and $\kappa = \beta^\beta$. Differentiating (6.2) with respect to y and defining

$$u(y,s) := \frac{d}{dy}v(y,s),$$

we get

$$\dot{u} = \frac{1}{\rho}\frac{d}{dy}\left(\rho\frac{d}{dy}u\right) - \frac{1}{2}u - \beta u + p(v+\kappa)^{p-1}u. \tag{6.3}$$

Next, computing the Taylor expansion of the nonlinear term, we write

$$p(v+\kappa)^{p-1}u = p\kappa^{p-1}u + p(p-1)(\kappa+\phi')^{p-2}vu = (\beta+1)u + c'(\phi',p)vu, \tag{6.4}$$

where ϕ' is between 0 and v and

$$c'(\phi',p) = p(p-1)(\kappa+\phi')^{p-2}.$$

Using (6.4) we can rewrite (6.3) as

$$\dot{u} = (\mathcal{L} - \frac{1}{2}I)u + c'(\phi',p)vu. \tag{6.5}$$

Working exactly as in Lemma 4.1 one can show:

Lemma 6.1 *For any $p > 1$*

$$0 \le c'(\phi',p) \le C',$$

where C' is a constant depending only on M and p.

Putting together (6.1) and (6.5) we arrive at the system

$$\dot{v} = \mathcal{L}v + c(\phi,p)v^2, \tag{6.6a}$$

$$\dot{u} = (\mathcal{L} - \frac{1}{2}I)u + c'(\phi',p)vu. \tag{6.6b}$$

Our plan is to work with system (6.6) using the techniques of Section 4. Notice that the linear operator $\mathcal{L} - \frac{1}{2}I$ of (6.6b) has eigenvalues 1/2, 0, -1/2, -1,... and therefore the unstable, neutral and stable subspaces are not the same as for the v-equation (6.6a). More specifically, the unstable subspace is now one dimensional, spanned by h_0, the neutral subspace is spanned by h_1, and the stable one is spanned by h_2 , $h_3, \ldots$, where h_0, h_1, h_2, ... are the eigenfunctions of $\mathcal{L} - I$ (see Section 2).

In analogy with the v-equation (Section 3) we define

$\tilde{\pi}_+$: orthogonal projection onto the span of h_0.
$\tilde{\pi}_0$: orthogonal projection onto the span of h_1.
$\tilde{\pi}_-$: orthogonal projection onto the span of h_2, $h_3, \ldots$

The following lemma is a direct consequence of (2.7).

Lemma 6.2 *Let $f(y) \in H^1_\rho$, and let π_* represent π_+, π_- or π_0. Then*

$$\frac{d}{dy}(\pi_* f) = \tilde{\pi}_*\left(\frac{d}{dy}f\right) \tag{6.7}$$

$$\|\pi_* f\|^2_{L^2_\rho} + \|\tilde{\pi}_*\left(\frac{d}{dy}f\right)\|^2_{L^2_\rho} = \|\pi_* f\|^2_{H^1_\rho}. \tag{6.8}$$

Proof: We will give the proof for π_-. (The other two cases are similar). Using (2.7) we write:

$$f = \sum_{j=0}^{\infty} \alpha_j h_j \quad \Rightarrow \quad \frac{d}{dy} f = \sum_{j=1}^{\infty} \alpha_j h_j' = \sum_{j=1}^{\infty} \alpha_j \left(\frac{j}{2}\right)^{1/2} h_{j-1}.$$

Hence

$$\tilde{\pi}_- \left(\frac{d}{dy} f\right) = \sum_{j=3}^{\infty} \alpha_j \left(\frac{j}{2}\right)^{1/2} h_{j-1}.$$

On the other hand:

$$\pi_- f = \sum_{j=3}^{\infty} \alpha_j h_j \quad \Rightarrow \quad \frac{d}{dy}(\pi_- f) = \sum_{j=3}^{\infty} \alpha_j \left(\frac{j}{2}\right)^{1/2} h_{j-1},$$

and (6.7) has been proved. Of course (6.8) is a direct consequence of (6.7).

□

Next we prove a result analogous to Theorem A, but with the H_ρ^1 norm replacing the L_ρ^2 norm.

Proposition 6.1 *Let $v(y,s)$ be a solution of (6.1) which does not decay exponentially fast. Then, given any $\epsilon > 0$ there exists an s_0 such that:*

$$\|v_+\|_{H_\rho^1} + \|v_-\|_{H_\rho^1} \le \epsilon \|v_0\|_{H_\rho^1}, \qquad for \quad s \ge s_0. \tag{6.9}$$

Proof: The proof is almost identical to that of Theorem A. Therefore we simply outline it, pointing out the differences.

Let $u_+ = \tilde{\pi}_+ u$ and similarly for u_0 and u_-. Recalling the notation of the previous sections we have that $x = \left(\int v_0^2 \rho\right)^{1/2}$, $y = \left(\int v_-^2 \rho\right)^{1/2}$, $z = \left(\int v_+^2 \rho\right)^{1/2}$, and $N = \left(\int v^4 \rho\right)^{1/2}$. From the L_ρ^2 theory (Section 4) we have for z that

$$\dot{z} \ge \frac{1}{2} z - CN. \tag{6.10}$$

Working similarly with equation (6.6b) we obtain for $z_1 = \left(\int u_+^2 \rho\right)^{1/2}$ that

$$\dot{z}_1 \ge \frac{1}{2} z_1 - C' N_1, \tag{6.11}$$

with $N_1 = \left(\int u^2 v^2 \rho\right)^{1/2}$. Let $z_2 = z + z_1$ and $N_2 = N + N_1$. Adding (6.10) and (6.11) we get

$$\dot{z}_2 \ge \frac{1}{2} z_2 - C_2 N_2, \tag{6.12}$$

where $C_2 = max(C, C')$. By repeating the above calculations for $x_2 = x + x_1$ and $y_2 = y + y_1$ with $x_1 = \left(\int u_0^2 \rho\right)^{1/2}$ and $y_1 = \left(\int u_-^2 \rho\right)^{1/2}$, we arrive at a system which is the analogue of (4.9):

$$\begin{aligned} \dot{z}_2 &\geq \frac{1}{2} z_2 - C_2 N_2 \\ |\, \dot{x}_2 \,| &\leq C_2 N_2 \\ \dot{y}_2 &\leq -\frac{1}{2} y_2 + C_2 N_2 \end{aligned} \tag{6.13}$$

We now estimate N_2. For N we already have (4.10):

$$N \leq \epsilon(x + y + z) + \delta^{\frac{k}{2}} J, \tag{6.14}$$

with $J = \left(\int v^4 \,|\, y\,|^k \rho\right)^{1/2}$. An argument similar to the proof of (4.10) gives:

$$\left(\int u^4 \rho\right)^{1/2} \leq \epsilon(x_1 + y_1 + z_1) + \delta^{\frac{k}{2}} J_1, \tag{6.15}$$

with $J_1 = \left(\int u^4 \,|\, y\,|^k \rho\right)^{1/2}$. We may thus estimate N_1 as follows:

$$N_1 = \left(\int u^2 v^2 \rho\right)^{1/2} \leq \left(\int v^4 \rho\right)^{1/4} \left(\int u^4 \rho\right)^{1/4} \leq \left(\int v^4 \rho\right)^{1/2} + \left(\int u^4 \rho\right)^{1/2}$$

$$\leq \epsilon(x + y + z) + \delta^{\frac{k}{2}} J + \epsilon(x_1 + y_1 + z_1) + \delta^{\frac{k}{2}} J_1,$$

Combining these estimates we conclude:

$$N_2 \leq 2\epsilon(x_2 + y_2 + z_2) + 2\delta^{\frac{k}{2}} J_2, \tag{6.16}$$

where $J_2 = J + J_1$.

We now estimate J_2. For J we have (4.18):

$$\dot{J} \leq -\frac{1}{2} J + \epsilon'(x + y + z). \tag{6.17}$$

For J_1 we multiply equation (6.6b) by $u^3 \,|\, y\,|^k \rho$ and integrate over all R. We then repeat step by step the calculations we did for J, from (4.11) through (4.18). (The fact that $u = v_y \to 0$ uniformly for $|\, y \,| \leq C$ as $s \to \infty$ follows from [13].) This leads to the estimate

$$\dot{J}_1 \leq -\frac{1}{2} J_1 + \epsilon'(x_1 + y_1 + z_1). \tag{6.18}$$

After adding (6.17) and (6.18) we conclude that

$$\dot{J}_2 \leq -\frac{1}{2} J_2 + \epsilon'(x_2 + y_2 + z_2). \tag{6.19}$$

The rest of the argument is the same as in the proof of Theorem A. We thus conclude finally that

$$z_2 + y_2 \leq \epsilon x_2, \qquad for \quad s \geq s_0. \tag{6.20}$$

Because of (6.8), (6.20) is equivalent to (6.9) and the Proposition has been proved.

□

Remark 6.1 Although the above proposition has been proved only for $n = 1$, an examination of the proof shows that the argument can be carried out for any space dimension $n \geq 1$ with minor changes.

□

We now give the proof of the uniform estimate.

Proof of Theorem C: Using the Sobolev embedding theorem we have

$$sup_{|y|<C} \mid v(y,s) - \frac{\kappa}{2ps}(1-\frac{1}{2}y^2) \mid \leq C'\|v(y,s) - \frac{\kappa}{2ps}(1-\frac{1}{2}y^2)\|_{H^1_\rho(|y|<C)} \leq$$

$$C'\|v_0(y,s) - \frac{\kappa}{2ps}(1-\frac{1}{2}y^2)\|_{H^1_\rho(|y|<C)} + C'\|v_+(y,s)\|_{H^1_\rho(|y|<C)} + C'\|v_-(y,s)\|_{H^1_\rho(|y|<C)} \leq$$

$$\leq C'\|v_0(y,s) - \frac{\kappa}{2ps}(1-\frac{1}{2}y^2)\|_{H^1_\rho} + C'\epsilon\|v_0(y,s)\|_{H^1_\rho} = O(\frac{\epsilon}{s}),$$

where we have used Propositions 5.1 and 6.1.

□

7 A Link Between Center Manifold Analysis and the Geometry of the Blowup Set.

We suggested in Section 2 that the local profile of the solution near blowup should reflect the geometry of the blowup set. A first step towards justifying this would be to prove that if the profile has a strict local maximum at $y = 0$ then the center of scaling is an isolated blowup point. We execute this here in the one dimensional case: our precise result is Theorem D.

Our argument makes use of the following result by Y. Giga and R. Kohn: if

$$E[w_a](s_0) < E[\kappa] \tag{7.1}$$

for some time s_0 then a is not a blowup point (see [13]). Here $E[w_a]$ denotes the "energy" functional, defined as follows for a solution of (1.3) rescaled about any point (a,T):

$$E[w_a](s) = \int e^{-\frac{y^2}{4}} \left(\frac{1}{2} \mid \nabla w_a(y,s) \mid^2 + \frac{1}{2(p-1)} w_a^2(y,s) - \frac{1}{p+1} w_a^{p+1}(y,s)\right) dy.$$

We will use this result in the following way. Assume that 0 is a blowup point. Recalling that $v = w - \kappa$, for $a \neq 0$ we shall obtain an expression of the form

$$E[w_a](s) = E[\kappa] + R(v)(s).$$

Then, using the asymptotic behavior of $v(y,s)$, we shall show that there is a time s_* after which $R(v)(s) < 0$, so that (7.1) holds. It follows that a is not a blowup point.

We now begin the argument in earnest. Suppose that 0 is a blowup point. Setting as usual

$$w(y,s) = (T-t)^{\frac{1}{p-1}}u(x,t), \quad y = \frac{x}{\sqrt{T-t}}, \quad s = -ln(T-t),$$

we have that $w(y,s) = v(y,s) + \kappa$, and the asymptotic behavior of $v(y,s)$ is known from the previous section. Let a be a point near 0. Rescaling about this point we have:

$$w_a(\tilde{y},s) = (T-t)^{\frac{1}{p-1}}u(x,t), \quad \tilde{y} = \frac{x-a}{\sqrt{T-t}}, \quad s = -ln(T-t).$$

We observe that $w_a(\tilde{y},s) = w(y,s)$ and $\tilde{y} = y - \frac{a}{\sqrt{T-t}} = y - \gamma$ with $\gamma = \frac{a}{\sqrt{T-t}}$.

The "energy" functional corresponding to w_a is thus

$$E[w_a](s) = \int e^{-\frac{\tilde{y}^2}{4}} \left(\frac{1}{2} \mid \nabla w_a(\tilde{y},s) \mid^2 + \frac{1}{2(p-1)} w_a^2(\tilde{y},s) - \frac{1}{p+1} w_a^{p+1}(\tilde{y},s)\right) d\tilde{y} =$$

$$= \int e^{-\frac{(y-\gamma)^2}{4}} \left(\frac{1}{2} \mid \nabla w(y,s) \mid^2 + F(w)\right) dy,$$

with $F(w) = \frac{1}{2(p-1)}w^2 - \frac{1}{p+1}w^{p+1}$. We next compute the Taylor expansion of $F(w)$ about $w = \kappa$. An easy calculation shows $F'(\kappa) = 0$ and $F''(\kappa) = -1$; therefore

$$F(w) = F(\kappa) - \frac{1}{2}(w-\kappa)^2 - \frac{1}{6}p(p-1)(\kappa+\xi)^{p-2}(w-\kappa)^3 = F(\kappa) - \frac{1}{2}v^2 - c(p,\xi)v^3,$$

with $c(p,\xi) = \frac{1}{6}p(p-1)(\kappa+\xi)^{p-2}$ and ξ between 0 and v. Moreover by the argument of Lemma 4.1 we have that $0 \le c(p,\xi) \le C''$ with C'' depending on M and p. Thus, we have

$$E[w_a](s) = E[\kappa] + \frac{1}{2}\int e^{-\frac{(y-\gamma)^2}{4}}(\mid \nabla v \mid^2 - v^2)dy - \int e^{-\frac{(y-\gamma)^2}{4}} c(p,\xi)v^3 dy. \tag{7.2}$$

Let

$$v^*(y,s) = \frac{\kappa}{2ps}(1 - \frac{1}{2}y^2).$$

We know that v behaves like v^* for large times (in the sense of Theorem C), so we rewrite (7.2) as

$$E[w_a](s) = E[\kappa] + \frac{1}{2}\int e^{-\frac{(y-\gamma)^2}{4}}(\mid \nabla v^* \mid^2 - v^{*2})dy$$

$$+\frac{1}{2}\int e^{-\frac{(y-\gamma)^2}{4}}(\mid \nabla v \mid^2 - \mid \nabla v^* \mid^2 - v^2 + v^{*2} - 2c(p,\xi)v^3)dy.$$

It follows that

$$E[w_a](s) = E[\kappa] - \frac{1}{2}\frac{\kappa^2}{4p^2s^2}\int e^{-\frac{(y-\gamma)^2}{4}}(1-2y^2+\frac{1}{4}y^4)dy + \int e^{-\frac{(y-\gamma)^2}{4}}\mathcal{E}(y,s)dy, \quad (7.3)$$

with

$$| \mathcal{E}(y,s) | \leq C(| v | + | v^* |)(| v - v^* |) + C(| \nabla v | + | \nabla v^* |)(| \nabla v - \nabla v^* |)$$
$$+ C | v |^3 := C(\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3), \quad (7.4)$$

for some positive constant C.

To prove Theorem D we will show:

(I) That the second term of (7.3) is negative (for γ fixed and different from 0) and therefore it behaves like $-\frac{C}{s^2}$ for suitable constant $C = C(\gamma)$ (depending only on γ) and sufficiently large time s.

(II) That the last term of (7.3) is of order ϵ/s^2 for arbitrarily small ϵ and sufficiently large s, i.e.

$$\int e^{-\frac{|y-\gamma|^2}{4}} | \mathcal{E}(y,s) | dy \leq \frac{\epsilon}{s^2}. \quad (7.5)$$

Once we have shown these, (7.3) will imply that after certain time s_*:

$$E[w_a](s) < E[\kappa].$$

Let $\rho_\gamma = e^{-\frac{|y-\gamma|^2}{4}}$. Working towards **(I)** we show

Lemma 7.1 *The function* $f(\gamma) = \int \rho_\gamma(1-2y^2+\frac{1}{4}y^4)dy$ *satisfies*

$$f(0) = 0, \quad and \quad f(\gamma) > 0 \quad for \quad \gamma \neq 0.$$

Proof: Clearly $f(\gamma) = f(-\gamma)$. By a simple integration by parts we find that $f(0) = 0$. An elementary calculation gives

$$f'(\gamma) = \int \rho_\gamma(-4y+y^3)dy, \qquad f'(0) = 0,$$

$$f''(\gamma) = \int \rho_\gamma(-4+3y^2), \qquad f''(0) = 2\int y^2\rho > 0,$$

$$f'''(\gamma) = 6\int \rho_\gamma y, \qquad f'''(0) = 0,$$

and finally $f''''(\gamma) = 6\int \rho_\gamma > 0$. We conclude that $f'''(\gamma), f''(\gamma), f'(\gamma), f(\gamma)$, are all positive for $\gamma > 0$.

□

The following lemma will be useful for establishing **(II)**. (We state it here but postpone the proof until later.)

Lemma 7.2 *In one space dimension* ($n = 1$), *given any* $\gamma \neq 0$ *and* $k \geq 2$, *for sufficiently large time we have*

$$\int_{y\cdot\gamma\geq 0} v^2 (y\cdot\gamma)^k \rho_\gamma dy \leq C^*/s^2, \tag{7.6}$$

with $C^* = C^*(\gamma, k)$ *depending on* γ *and* k.

To prove **(II)** we now examine the large time behavior of each of the $\int \mathcal{E}_i \rho_\gamma$ separately. We start with $\int \mathcal{E}_1 \rho_\gamma$:

$$\int \mathcal{E}_1(y,s)dy \leq \left(\int (|\, v \,| + |\, v^* \,|)^2 \rho_\gamma dy\right)^{1/2} \left(\int (v - v^*)^2 \rho_\gamma dy\right)^{1/2}. \tag{7.7}$$

Clearly

$$\int v^{*2} \rho_\gamma \leq C/s^2,$$

whereas fixing some $\mu > 0$ we have that

$$\int v^2 \rho_\gamma = \int_{y\cdot\gamma\leq\mu^{-1}} v^2 \rho_\gamma + \int_{y\cdot\gamma\geq\mu^{-1}} v^2 \rho_\gamma \leq e^{\frac{\mu^{-1}}{2} - \frac{\gamma^2}{4}} \int v^2 \rho + \mu^2 \int_{y\cdot\gamma\geq 0} v^2 (y\cdot\gamma)^2 \rho_\gamma \leq \frac{C}{s^2},$$

where we used Lemma 7.2. Thus we have

$$\left(\int (|\, v \,| + |\, v^* \,|)^2 \rho_\gamma dy\right)^{1/2} \leq C/s. \tag{7.8}$$

We still have to show that

$$\int (v - v^*)^2 \rho_\gamma \leq \epsilon^2/s^2, \tag{7.9}$$

and this is the most technical estimate. To show that (7.9) is true we write (for any $\delta > 0$):

$$\int (v - v^*)^2 \rho_\gamma \leq \int_{|y|\leq\delta^{-1}} (v - v^*)^2 \rho_\gamma + 2 \int_{|y|\geq\delta^{-1}} v^2 \rho_\gamma + 2 \int_{|y|\geq\delta^{-1}} v^{*2} \rho_\gamma. \tag{7.10}$$

For the first term of (7.10) we have that

$$\int_{|y|\leq\delta^{-1}} (v - v^*)^2 \rho_\gamma \leq C(\delta) \int_{|y|\leq\delta^{-1}} (v - v^*)^2 \rho \leq \frac{\epsilon' C(\delta)}{s^2},$$

using Theorem C. For the second term of (7.10) we write, for any $\mu > 0$:

$$\int_{|y|\geq\delta^{-1}} v^2 \rho_\gamma = \int_{\{|y|\geq\delta^{-1}\}\cap\{y\cdot\gamma\leq\mu^{-1}\}} v^2 \rho_\gamma + \int_{\{|y|\geq\delta^{-1}\}\cap\{y\cdot\gamma\geq\mu^{-1}\}} v^2 \rho_\gamma$$

$$\leq C(\mu)\delta^k \int v^2 \,|\, y \,|^k \rho + \mu^k \int_{y\cdot\gamma\geq 0} v^2 (y\cdot\gamma)^k \rho_\gamma \leq C(\mu) c_0(k) \delta^4 \int v_0^2 \rho + \mu^k \frac{C^*}{s^2}$$

In the above estimates we used Lemmas 5.1 and 7.2. We thus have

$$\int_{|y|\geq\delta^{-1}} v^2 \rho_\gamma \leq \left(C(\mu) c_0(k) \delta^4 + \mu^k C^*\right) \frac{1}{s^2}. \tag{7.11}$$

Finally, for the last term of (7.10) , using the fact that all norms of v^* are equivalent we get

$$\int_{|y|\geq\delta^{-1}} v^{*2}\rho_\gamma \leq \delta^k \int v^{*2} \mid y \mid^k \rho_\gamma \leq \delta^k \frac{C'(k,\gamma)}{s^2}.$$

Combining the above we end up with

$$\int (v - v^*)^2 \rho_\gamma \leq \left(\epsilon' C(\delta) + C(\mu) c_0(k) \delta^4 + \delta^k C'(k,\gamma) + \mu^k C^*\right) \frac{1}{s^2},$$

and assertion (7.9) follows by making a suitable choice of μ, δ and ϵ'. Notice that one has to choose μ first, then δ , and finally ϵ'. From (7.7)-(7.9) we conclude that given any ϵ there exists a time s_1 after which

$$\int e^{-\frac{(y-\gamma)^2}{4}} \mathcal{E}_1(y,s)dy \leq \frac{\epsilon}{s^2}. \tag{7.12}$$

Estimating $\mathcal{E}_2$ in exactly the same way we conclude that after certain time s_2 we have

$$\int e^{-\frac{(y-\gamma)^2}{4}} \mathcal{E}_2(y,s)dy \leq \frac{\epsilon}{s^2}. \tag{7.13}$$

To show (7.13) one needs the analogues of Lemmas 5.1 and 7.2 for ∇v. The proofs of these are the same as for v.

We finally estimate the term $\int \mathcal{E}_3 \rho_\gamma$ as follows:

$$\int \mid v \mid^3 \rho_\gamma = \int_{|y|\leq\delta^{-1}} \mid v \mid^3 \rho_\gamma + \int_{|y|\geq\delta^{-1}} \mid v \mid^3 \rho_\gamma \leq C(\delta) \int_{|y|\leq\delta^{-1}} \mid v \mid^3 \rho + M \int_{|y|\geq\delta^{-1}} \mid v \mid^2 \rho_\gamma,$$

where we used the fact that $\mid v \mid \leq M$. Recalling that $v \to 0$ uniformly on compact sets and using (7.11) we have

$$\int \mid v \mid^3 \rho_\gamma \leq \epsilon' C(\delta) \int_{|y|\leq\delta^{-1}} \mid v \mid^2 \rho + M \left(C(\mu) c_0(k) \delta^4 + \mu^k C^*\right) \frac{1}{s^2}$$

$$\leq \epsilon' C(\delta) \frac{C}{s^2} + M \left(C(\mu) c_0(k) \delta^4 + \mu^k C^*\right) \frac{1}{s^2}.$$

By choosing μ, δ, ϵ' suitably we can arrange that after certain time s_3

$$\int e^{-\frac{(y-\gamma)^2}{4}} \mathcal{E}_3(y,s)dy \leq \frac{\epsilon}{s^2}. \tag{7.14}$$

Thus, we have shown that after $s_* = max\{s_1,\ s_2,\ s_3\}$

$$\int e^{-\frac{(y-\gamma)^2}{4}} \mid \mathcal{E}(y,s) \mid dy \leq \frac{\epsilon}{s^2}. \tag{7.15}$$

It follows using (7.3) that after s_*

$$E[w_a](s) < E[\kappa]. \tag{7.16}$$

We conclude that a is not a blowup point. Since $a = \gamma\sqrt{T-t}$ and $T - t \to 0$ as $s \to \infty$, the possible values of a fill out a whole neighborhood of 0. The proof of Theorem D is now complete.

We have yet to give the proof of Lemma 7.2. It is a consequence of the following result, which holds in any space dimension $n \geq 1$. For $\gamma \in R^n$, we set $\rho_\gamma(y) = e^{-\frac{|y-\gamma|^2}{4}}$, $H_\gamma = \{y \cdot \gamma \geq 0\}$ and

$$I_\gamma(s) = \int_{H_\gamma} v^2 (y \cdot \gamma)^k \rho_\gamma dy.$$

Lemma 7.3 *For any $\gamma \neq 0$ and $k \geq 2$ there exists a time s_0 after which*

$$\frac{d}{ds} I_\gamma(s) \leq -I_\gamma(s) + C(\gamma, k) \int v^2 \rho\,, \tag{7.17}$$

where $C(\gamma, k)$ is a positive constant depending only on γ and k.

Proof: We calculate $\frac{d}{ds} I_\gamma$ using equation (4.5) for v. Multiplying (4.5) with $v(y \cdot \gamma)^k \rho_\gamma$ and integrating over H_γ we get

$$\frac{1}{2}\frac{d}{ds} I_\gamma = \int_{H_\gamma} \nabla(\rho \nabla v) \frac{\rho_\gamma}{\rho} v (y \cdot \gamma)^k + \int_{H_\gamma} v^2 (y \cdot \gamma)^k \rho_\gamma + \int_{H_\gamma} c(\phi, p) v^3 (y \cdot \gamma)^k \rho_\gamma. \tag{7.18}$$

Integration by parts in the first term (we choose $k \geq 2$ so that all boundary terms cancel in the integration by parts) gives, after some calculation,

$$\int_{H_\gamma} \nabla(\rho \nabla v) \frac{\rho_\gamma}{\rho} v (y \cdot \gamma)^k \leq \frac{k}{2} \int_{H_\gamma} div(\rho_\gamma (y \cdot \gamma)^{k-1} \gamma) v^2 + \frac{1}{4} \int_{H_\gamma} div(\rho_\gamma (y \cdot \gamma)^k \gamma) v^2.$$

Now,

$$div\left(\rho_\gamma (y \cdot \gamma)^{k-1} \gamma\right) = -\frac{1}{2}(y \cdot \gamma)^k \rho_\gamma + \frac{|\gamma|^2}{2}(y \cdot \gamma)^{k-1} \rho_\gamma + |\gamma|^2 (k-1)(y \cdot \gamma)^{k-2} \rho_\gamma,$$

and

$$div\left(\rho_\gamma (y \cdot \gamma)^k \gamma\right) = -\frac{1}{2}(y \cdot \gamma)^{k+1} \rho_\gamma + \frac{|\gamma|^2}{2}(y \cdot \gamma)^k \rho_\gamma + |\gamma|^2 k (y \cdot \gamma)^{k-1} \rho_\gamma.$$

Therefore, we conclude that

$$\int_{H_\gamma} \nabla(\rho \nabla v) \frac{\rho_\gamma}{\rho} v (y \cdot \gamma)^k \leq \int_{H_\gamma} v^2 \mathcal{S} \rho_\gamma, \tag{7.19}$$

with

$$\mathcal{S} = -\frac{1}{8}(y \cdot \gamma)^{k+1} + \left(\frac{|\gamma|^2}{8} - \frac{k}{4}\right)(y \cdot \gamma)^k + \frac{k|\gamma|^2}{2}(y \cdot \gamma)^{k-1} + \frac{k(k-1)|\gamma|^2}{2}(y \cdot \gamma)^{k-2}.$$

The last two terms of (7.18) are dominated by $(1+CM)I_\gamma$ since $c(\phi,p)v \leq CM$. Thus, from (7.18) we have

$$\frac{d}{ds}\int_{H_\gamma} v^2(y\cdot\gamma)^k\rho_\gamma dy \leq \int_{H_\gamma} v^2\left[2\mathcal{S}+(2+2CM)(y\cdot\gamma)^k\right]\rho_\gamma dy. \tag{7.20}$$

Now, if $y\cdot\gamma \gg 1$ then the term $-\frac{1}{8}(y\cdot\gamma)^{k+1}$ dominates in $\mathcal{S}$ and $2\mathcal{S}+(2+2CM)(y\cdot\gamma)^k \leq -(y\cdot\gamma)^k$. Thus there is a constant $c_2 = c_2(\gamma,k)$ such that

$$\frac{d}{ds}I_\gamma \leq -I_\gamma + C\int_{0\leq y\cdot\gamma\leq c_2} v^2\rho_\gamma,$$

for some constant C depending on γ and k. Since

$$\rho_\gamma \leq e^{\frac{c_2}{2}-\frac{|\gamma|^2}{4}}\rho, \quad for \quad y\cdot\gamma\leq c_2,$$

we finally conclude that

$$\frac{d}{ds}I_\gamma \leq -I_\gamma + C(\gamma,k)\int v^2\rho,$$

and assertion (7.17) has been proved.

□

We now give the proof of Lemma 7.2. In the one dimensional case we know that for sufficiently large s

$$\int v^2\rho \leq C/s^2,$$

for some C. Therefore from (7.17) we have

$$\frac{d}{ds}I_\gamma \leq -I_\gamma + C/s^2. \tag{7.21}$$

Integrating (7.21) from (some fixed time) s_1 to s we obtain

$$I_\gamma(s) \leq I_\gamma(s_1)e^{-s} + C\int_{s_1}^{s}\frac{e^{s'-s}}{s'^2}ds'. \tag{7.22}$$

Since $\int_{s_1}^{s}\frac{e^{s'-s}}{s'^2}ds' \leq C'/s^2$, for suitable constant C', Lemma 7.2 has been proved.

□

Remark 7.1 . A two dimensional generalization of Theorem D will be presented in [7]. We indicate briefly the nature of this extension. The first step is to justify the formal analysis of Section 2, leading to the conclusion that either

$$v(y,s) \approx \frac{\kappa}{ps}(1-\frac{1}{4}|y|^2) \tag{7.23}$$

or else

$$v(y,s) \approx \frac{\kappa}{2ps}(1-\frac{1}{2}(y\cdot\eta)^2). \tag{7.24}$$

The second step is an argument similar to the one presented in this section. When (7.23) holds, it shows that the center of scaling is an isolated blowup point. When (7.24) holds, it shows instead that the blowup set lies inside a cone of arbitrarily small aperture, based at the center of scaling and containing the line orthogonal to η.

8 Future Directions

We close by discussing the current state of this theory, taking into account the results in [16-18, 23] as well as those presented here.

Space Dimension One

Our analysis depends on the hypothesis that v does not tend to zero exponentially fast. Formally, this is the generic case: it should be true so long as the initial data avoid a manifold in function space of codimension 1. The only rigorous result concerning this hypothesis is due to Herrero and Velazquez: they prove that it holds whenever the initial condition is unimodal [16].

We noted in Section 2 that if v tends to zero exponentially fast, then its profile should resemble one of the higher eigenfunctions $h_3(y)$, $h_4(y), \ldots$. Herrero and Velazquez prove in [17] that only the even eigenfunctions $h_{2k}(y)$ can occur, and they show that h_4 does occur. One expects that solutions should exist with profile $h_{2k}(y)$ for any $k = 2, 3, 4, \ldots$, but this remains open.

Higher Space Dimensions

In space dimension two, the next task is to justify rigorously the formal results (2.32), (2.33). This has been accomplished, and will be presented in [7].

Even the formal picture is incomplete in higher space dimensions. An analysis similar to Lemma 2.3 has not been done for space dimensions $n \geq 3$. The situation when $v \to 0$ exponentially fast is unclear even for $n = 2$ (see the remarks at the end of Section 2).

We believe that the local character of the blowup set is determined by the asymptotic profile of v. The method of Section 7 shows, roughly speaking, that if v does not tend to zero exponentially fast, then there can be no blowup where $v(y, s) << -\frac{1}{s}$. The full power of the method remains unclear. Perhaps it could be used to prove that the blowup set has Hausdorff dimension at most $n - 1$ in R^n . Thus far there is no known restriction on the size of the blowup set.

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