Fast blow up mechanisms for sign-changing solutions of a semilinear parabolic equation with critical nonlinearity

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Abstract

We consider problem (1.1), (1.2) below. Using formal arguments based on matched asymptotic expansion techniques, we give a detailed description of radially symmetric, sign-changing solutions, which blow up at $x = 0$ and $t = T < \infty$, for space dimension $N = 3, 4, 5, 6$. These solutions exhibit fast blow up, that is, they satisfy: $\lim_{t\uparrow T}(T-t)^{\frac{1}{p-1}}u(0,t) = \infty$. In contrast, radial solutions that are positive and decreasing behave as in the subcritical case for any $N \geq 3$. This last result is extended in the case of exponential nonlinearity and $N = 2$.

1 Introduction

The purpose of this work is to describe at a formal level some mechanisms of singularity formation for the semilinear parabolic equation:

$$u_t = \Delta u + |u|^{p-1}u; \qquad u(x,0) = u_0(x), \qquad x \in R^N, \qquad t > 0, \tag{1.1}$$

with critical power nonlinearity, that is:

$$p = \frac{N+2}{N-2}, \qquad\qquad N \geq 3. \tag{1.2}$$

The unfolding of singularities in solutions of evolution equations is a subject of both practical and theoretical interest, and has been extensively dealt with in the literature. References include analysis of high activation energy asymptotics in combustion theory (Blythe & Crighton 1989, Dold 1985), shrinking of surfaces evolving by mean curvature flow (Angenent & Velázquez 1995, Gage & Hamilton 1987), aggregation phenomena in microorganism colonies (Keller & Segel 1970, Herrero & Velázquez 1996), breakup of free surface flows (Eggers 1997), and pinchoff of droplets under motion by surface diffusion (Bernoff *et al* 1998), to mention but a few.

Because of its apparent simplicity, problem (1.1), has been widely considered as a testfield for methods intended to unravel the behaviour of solutions near singularity formation. Indeed, it is a well known fact that solutions of (1.1) with $p > 1$, may blow

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up (i.e. become unbounded) in finite time, even if they start from bounded and smooth data at $t = 0$. Currently there exist rather detailed descriptions of the asymptotics of nonnegative blowing up solutions of (1.1) in the case of subcritical nonlinearities, that is, when $1 < p < \frac{N+2}{N-2}$ if $N \geq 3$ or $p > 1$ if $N = 1, 2$; cf. Giga & Kohn (1985, 1987), Filippas & Kohn (1992), Velázquez (1992, 1993). However, little seems to be known when condition (1.2) holds.

Let us denote by T the time at which the solution becomes unbounded. A crucial first step towards the study of $u(x,t)$ near blow up is the following estimate that was derived by Giga & Kohn (1987) for $u_0(x) \geq 0$ and $1 < p < \frac{N+2}{N-2}$:

$$|u(x,t)| \leq C(T-t)^{-\frac{1}{p-1}} \quad \text{for some } \ C > 0 \quad \text{and} \quad 0 < t < T. \tag{1.3}$$

Notice that (1.3) is a natural upper bound, since it corresponds to the behaviour exhibited by solutions of the first order PDE:

$$u_t = |u|^{p-1}u,$$

that can be explicitly integrated along characteristics. As a matter of fact, it used to be widely assumed that (1.3) should hold for all solutions of (1.1), regardless of the precise values of the parameter $p > 1$ and the space dimension N. It was therefore a bit surprising to find that (1.3) may actually fail when p and N are large enough and $u_0(x) \geq 0$, as shown by Herrero & Velázquez (1994). When (1.3) fails we say that there is fast blow up.

On the other hand, it is worth to be mentioned that estimate (1.3) has been obtained by Giga & Kohn (1987), even in the case of sign changing solutions, under the rather strange restriction that $1 < p < \frac{3N+8}{3N-4}$ if $N \geq 2$ or $p > 1$ if $N = 1$. Actually, it is not clear whether there exist sign changing solutions of (1.1) with $p > \frac{3N+8}{3N-4}$, for which (1.3) fails. The purpose of this work is to show that sign changing solutions of (1.1) exhibiting fast blow up do exist, if we take p as in (1.2).

More precisely we provide a rather detailed description of special solutions of problem (1.1), (1.2), for $N = 3, 4, 5, 6$ that satisfy:

$$\lim_{t \to T^-} (T-t)^{\frac{1}{p-1}} u(0,t) = +\infty. \tag{1.4}$$

A remarkable feature of the solutions we construct is that, although they are unbounded near the singularity as $t \to T$ with $t < T$, they are bounded near the blow up point (in our case the origin) at $t = T$.

Since the critical exponent $p = \frac{N+2}{N-2}$ is not covered by the results of Giga & Kohn (1987), it is natural to wonder if it would be possible to obtain nonnegative solutions of (1.1), (1.2) for which (1.3) fails. We provide a (partial) negative answer to this question, since we prove that nonnegative monotonically decreasing radial solutions of (1.1), (1.2) satisfy (1.3). Let us note that this is in clear contrast with the situation for large values of p and N, described by Herrero & Velázquez (1994), where the existence of radial positive and decreasing fast blowing up solutions is shown. For completeness we also show that there is no fast blow up in the two dimensional case, when the power nonlinearity in (1.1) is replaced by e^u.

The method we use is based on matched asymptotic expansions techniques. A crucial idea consists in deriving a suitable equation for a characteristic length that, roughly speaking, describes the size of an inner layer where solutions grow unbounded. Solving this equation eventually allows us to derive the precise asymptotics of solutions as blow up is approached.

Once the desired behaviours have been identified, a rigorous proof can be provided by means of suitable topological fixed point arguments. At the technical level, this last is by no means a simple task (see for instance Herrero & Velázquez (1996, 1997) for examples of application of this method in other types of problems). To keep this work within reasonable bounds, we have decided to focus here in the question of deriving the most essential information concerning the blow up profiles. A rigorous proof corresponding to the case $N = 3$ will appear elsewhere.

We conclude this Introduction by describing the plan of the paper. A few preliminary results are gathered in Section 2 below. Section 3 is then concerned with describing the blow up mechanism corresponding to space dimension $N = 3$. The case $N = 4$ makes the content of Section 4, whereas dimensions $N \geq 5$ are dealt with in Section 5. The behaviours described in these Sections correspond to solutions which are radial and exhibit changes of sign. Positive and radially decreasing solutions are then examined in Section 6. A short Section 7 is then devoted to the case of exponential nonlinearity appearing in the reaction term. The paper then concludes with three Appendices: In Appendix A we recall some properties of Laguerre and Hermite polynomials. In Appendix B we derive the asymptotics of an auxiliary function appearing in Section 3. Finally, some (indicative) drawings of the blow up structures derived in Sections 3-5 are presented in Appendix C.

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2 Preliminaries

In order to study the blowing up solutions of (1.1) we introduce, as usual, similarity variables as follows:

$$y = x(T-t)^{-1/2}, \qquad \tau = -\log(T-t), \qquad \Phi(y,\tau) = (T-t)^{\frac{1}{p-1}} u(x,t).$$

In this set of variables Φ exists for all times $\tau > 0$ and satisfies the equation:

$$\Phi_\tau = \Delta\Phi - \frac{y \cdot \nabla\Phi}{2} - \frac{\Phi}{p-1} + |\Phi|^{p-1}\Phi. \tag{2.1}$$

The linear operator

$$A \equiv \Delta - \frac{y \cdot \nabla}{2} - \frac{I}{p-1},$$

has been repeatedly used in the analysis of blowing up solutions of (1.1). It is defined (and is self-adjoint) in the Hilbert space:

$$L^2_w(I\!R^N) = \{f \in L^2_{loc}(I\!R^N): \quad \|f\|^2 = \int_{I\!R^N} |f(y)|^2 e^{-|y|^2/4} d^N y < \infty\}.$$

We denote the inner product in this space by:

$$< f, g > = \int_{I\!R^N} f(y) g(y) e^{-|y|^2/4} d^N y. \tag{2.2}$$

The Sobolev spaces $H^k_w(I\!R^N)$, $k = 1, 2, \ldots$, are then defined in the usual way (cf. Herrero & Velázquez 1997). We restrict our analysis to radial functions, and hence it is natural to take as domain of the operator A the linear space:

$$D(A) = \{f \in L^2_w(I\!R^N): \quad f(x) = f(|x|) \;\text{ for a.e. }\; x \in I\!R^N\} \cap H^2_w(I\!R^N).$$

By standard results it follows easily that A has a discrete sequence of eigenvalues given by:

$$\lambda_k = -k - \frac{1}{p-1}, \qquad k = 0, 1, 2, \ldots.$$

The corresponding eigenfunctions are given by:

$$\phi_k(y) = C_{k,N} L_k^{(\frac{N-2}{N})}(\frac{y^2}{4}), \qquad k = 0, 1, 2 \ldots,$$

where $L_k^{(a)}(x)$ stands for the modified Laguerre polynomials and the constants $C_{k,N}$ are so chosen in order to normalize the ϕ_k's, so that $\|\phi_k\| = 1$, for any k; cf Appendix A.

The key idea in the mechanism of singularity formation that we will describe here is the following. Whereas for subcritical p (i.e. when $1 < p < \frac{N+2}{N-2}$) equation (1.1) has no positive bounded steady states (cf. Gidas & Spruck 1981), in contrast, for the critical value $p = \frac{N+2}{N-2}$ equation (1.1) admits a positive steady solution of the form:

$$\bar{u}(x) = \left(1 + \frac{|x|^2}{N(N-2)}\right)^{-\frac{2}{p-1}} = \left(1 + \frac{|x|^2}{N(N-2)}\right)^{-\frac{N-2}{2}}, \tag{2.3}$$

and it is possible to obtain a one-parameter family of steady solutions of (1.1) by means of the rescaling $\bar{u}_\lambda(x) = \lambda^{-\frac{2}{p-1}} \bar{u}_\lambda(\frac{x}{\lambda})$. Notice that equation (2.1) resembles strongly the original equation (1.1) if $|y| \ll 1$ and Φ is large, as can be seen from dimensional arguments. *The key idea then is to assume that* Φ *behaves as* $\bar{u}_{\lambda(\tau)}(y)$ *in the region* $|y| \ll 1$ *for some* $\lambda(\tau) \to 0$. We remark that the relation between the upper bound on the blowup rate (1.2) and the absence of steady states for (1.1) was already noticed by Giga & Kohn (1987). On the other hand, the construction in Herrero & Velázquez (1994) of solutions that blowup with a rate faster than (1.2) for large values of p and N, was based in an essential way on the existence of stable steady states of (1.1), an idea which has also been pursued in Herrero & Velázquez (1996).

Consider now the stationary version of (1.1), namely:

$$\Delta u + |u|^{p-1}u = 0.$$

When linearizing in that equation about its solution $\bar{u}$, we obtain:

$$\Delta w + p\bar{u}^{p-1}w = 0. \tag{2.4}$$

A simple calculation shows that a solution of (2.4) is given by:

$$\bar{\phi}_0(x) = \frac{x \cdot \nabla \bar{u}(x)}{2} + \frac{\bar{u}(x)}{p-1}.$$

After a straightforward calculation we find that:

$$\bar{\phi}_0(x) = \left(1 + \frac{|x|^2}{N(N-2)}\right)^{-\frac{N}{2}} \left(\frac{N-2}{4} - \frac{|x|^2}{4N}\right). \tag{2.5}$$

From (2.3) and (2.5) we easily derive the following asymptotics:

$$\begin{aligned} \bar{u}(x) \quad &\sim \quad (N(N-2))^{\frac{N-2}{2}}|x|^{-(N-2)}\left(1 + O(\frac{1}{|x|^2})\right), \qquad x \to +\infty, \\ \bar{\phi}_0(x) \quad &\sim \quad -\frac{(N(N-2))^{\frac{N}{2}}}{4N}|x|^{-(N-2)}\left(1 + O(\frac{1}{|x|^2})\right), \qquad x \to +\infty. \end{aligned}$$

We are now ready to begin our study, starting from the case $N = 3$.

3 Space dimension N=3

Here we will present the arguments leading to the results in the case $N = 3$. Since many of the ideas to be described are the same even when $N \geq 4$, we will not use the specific values $N = 3$, $p = 5$ until we really need to do so. On the other hand, we shall only deal with radial solutions. For notational convenience, however, we shall continue to use the symbols Δ, ∇, ... instead of their radial counterparts.

As we mentioned earlier, we will assume that Φ is approximated by the solution of the steady state (cf (2.3)), suitably rescaled, when $|y| \ll 1$. Let us denote by $\varepsilon(\tau)$ a characteristic length, which is a priori unknown, that will be a measure of the size of the region where Φ can be approximated in such a way. In order to understand what happens for $|y| \ll 1$, we introduce a new set of variables (*inner variables*) as follows:

$$\xi = y/\varepsilon(\tau), \qquad \Phi(y, \tau) = \varepsilon^{-\frac{2}{p-1}}(\tau)G(\xi, \tau). \tag{3.1}$$

Using (3.1) in (2.1), we obtain:

$$\varepsilon^2(\tau)G_\tau = \Delta G + |G|^{p-1}G + \sigma(\tau)\left(\frac{\xi\cdot\nabla G}{2} + \frac{G}{p-1}\right), \tag{3.2}$$

with:

$$\sigma(\tau) \equiv 2\varepsilon(\tau)\dot{\varepsilon}(\tau) - \varepsilon^2(\tau),$$

and where all spatial derivatives are computed with respect to the new variable ξ.

By assumption $\varepsilon(\tau) \to 0$ as $\tau \to +\infty$. We will also assume that $|\dot{\varepsilon}(\tau)| \le C\varepsilon(\tau)$, whence $\sigma(\tau) \to 0$, as $\tau \to +\infty$. It is then natural to expect that solutions of (3.2) will behave to the lowest order as:

$$G(\xi,\tau) \sim \bar{u}(\xi), \qquad \text{as } \ \tau \to +\infty. \tag{3.3}$$

Notice that in principle we could only expect that $G(\xi,\tau)$ would approach to a steady state of (1.3) (as we have seen in Section 2, there is a one parameter family of them). It is possible, however, to select the particular function $\bar{u}(\xi)$ in (2.3) by changing if needed the definition of $\varepsilon(\tau)$. In fact we can assume that $\varepsilon(\tau)$ is defined by means of the formula:

$$\Phi(0,\tau) = \varepsilon^{-\frac{2}{p-1}}(\tau). \tag{3.4}$$

We need to compute the next order correction to $G(\xi,\tau)$ (cf (3.3)). To this end we introduce a new function ϕ defined by:

$$\phi(\xi,\tau) = G(\xi,\tau) - \bar{u}(\xi). \tag{3.5}$$

Notice that (3.4) implies that:

$$\phi(0,\tau) = 0. \tag{3.6}$$

On the other hand, formally linearising (3.2) about $\bar{u}$ we obtain to the lowest order:

$$\varepsilon^2\phi_\tau = \Delta\phi + p\bar{u}^{p-1}\phi + \sigma(\tau)\left(\frac{\xi\cdot\nabla\bar{u}}{2} + \frac{\bar{u}}{p-1}\right). \tag{3.7}$$

By assumption $|\phi| \to 0$, as $\tau \to +\infty$. We then expect from (3.7) the following asymptotics for ϕ:

$$\phi(\xi,\tau) \sim \sigma(\tau)H(\xi), \qquad \text{as } \ \tau \to +\infty, \tag{3.8}$$

where $H(\xi)$ solves the equation:

$$\Delta H + p\bar{u}^{p-1}H + \left(\frac{\xi\cdot\nabla\bar{u}}{2} + \frac{\bar{u}}{p-1}\right) = 0, \tag{3.9}$$

complemented with the following initial conditions:

$$H(0) = 0, \qquad H'(0) = 0.$$

The first condition follows from (3.6), whereas the second one is a consequence of the radial symmetry of the solutions under consideration.

So far, no choice of the space dimension N has been made. We now make use of the special values $N = 3$, $p = 5$. The reason for doing so, is that the asymptotic behaviour of H, that will play an important role in the sequel, depends crucially on the dimension N. In the three dimensional case the asymptotics of H is given by (cf Appendix B):

$$H(\xi) \sim \frac{\sqrt{3}}{8}\xi + O(1), \qquad \text{as } |\xi| \to +\infty.$$

Taking into account (3.5), (3.8) we obtain the following expansion for G:

$$G(\xi, \tau) \sim \bar{u}(\xi) + \sigma(\tau)H(\xi) + \dots.$$

Using the asymptotics of $\bar{u}(\xi)$ and $H(\xi)$ for $|\xi| \gg 1$ we have:

$$G(\xi, \tau) \sim \frac{\sqrt{3}}{|\xi|} + \frac{\sqrt{3}\sigma(\tau)}{8}|\xi| + \dots, \qquad \text{as } |\xi| \to +\infty, \qquad \tau \to +\infty.$$

Returning then to the original variables (cf (3.1)) we obtain the following behaviour:

$$\Phi(y, \tau) \sim \frac{\sqrt{3}\varepsilon^{1/2}(\tau)}{|y|} + \frac{\sqrt{3}\sigma(\tau)}{8\varepsilon^{3/2}(\tau)}|y| + \dots, \qquad \text{as } \tau \to +\infty, \qquad \varepsilon(\tau) \ll |y| \ll 1. \tag{3.10}$$

It follows from its definition that $\sigma(\tau)$ is roughly of order $\varepsilon^2(\tau)$. The expansion (3.10) then suggests that Φ becomes of order $\varepsilon^{1/2}(\tau)$ in the region $|y| \sim 1$, whence it is very small there. This observation also hints at neglecting the term $|\Phi|^{p-1}\Phi$ in (2.1) (away from the origin) since $p > 1$. It is clear, however, that we cannot discard it in the region close to $\xi = 0$. It is then natural to replace $|\Phi|^{p-1}\Phi$ in (2.1) by a term concentrated near the origin. More precisely, we set:

$$|\Phi(y, \tau)|^{p-1}\Phi(y, \tau) \sim c(\tau)\delta(y), \qquad \text{as } \tau \to +\infty,$$

where $\delta(y)$ is a Dirac mass located at the origin and $c(\tau)$ denotes its strength. This last is uniquely determined by requiring that:

$$\int_{I\!R^N} |\Phi(y, \tau)|^{p-1}\Phi(y, \tau)d^N y = \int_{I\!R^N} c(\tau)\delta(y)d^N y = c(\tau).$$

The term in the left hand side can be estimated in the following manner. Taking into account (3.1), (3.3) and (1.3) we write:

$$\begin{aligned}\int_{I\!R^N} |\Phi(y, \tau)|^{p-1}\Phi(y, \tau)d^N y &\sim \varepsilon^{-\frac{2p}{p-1}} \int_{I\!R^N} \bar{u}^p(\frac{y}{\varepsilon})d^N y = \varepsilon^{\frac{N-2}{N}} \int_{I\!R^N} \bar{u}^p(\xi)d^N \xi \\ &= -\varepsilon^{\frac{N-2}{N}} \lim_{R\to+\infty} \int_{|\xi|\le R} \Delta\bar{u}(\xi)d^N\xi = \varepsilon^{\frac{N-2}{N}} \lim_{R\to+\infty} |\nabla\bar{u}(R)|K_{N-1}R^{N-1}.\end{aligned}$$

where $K_{N-1} = 2\pi^{N/2}/\Gamma(N/2)$ is the area of the unit sphere in $I\!R^N$, and $\Gamma(z)$ denotes here the standard Euler's function. Since $\bar{u}$ is explicit (cf (2.3)) we compute:

$$|\nabla\bar{u}(R)| \sim (N-2)^{\frac{N}{2}} N^{\frac{N-2}{2}} R^{-(N-1)}, \qquad R \to +\infty.$$

Thus, we finally have:

$$c(\tau) = \frac{2\pi^{\frac{N}{2}}(N-2)^{\frac{N}{2}}N^{\frac{N-2}{2}}}{\Gamma(\frac{N}{2})}\varepsilon^{\frac{N-2}{2}}(\tau). \tag{3.11}$$

In the case at hand where $N = 3$, we readily see that:

$$c(\tau) = 4\pi\sqrt{3}\varepsilon^{1/2}(\tau).$$

We are thus led to the following linear approximation of (2.1), valid for regions away from the origin, that is when $|y| \gg \varepsilon(\tau)$:

$$\Phi_\tau = \Delta\Phi - \frac{y\cdot\nabla\Phi}{2} - \frac{\Phi}{4} + 4\pi\sqrt{3}\varepsilon^{1/2}(\tau)\delta(y), \qquad \tau \gg 1. \tag{3.12}$$

Notice that the approximation (3.12) is rather natural, in the sense that the leading term in the right hand side of (3.10) clearly indicates the presence of a Dirac mass at the origin with strength exactly equal to $4\pi\sqrt{3}\varepsilon^{1/2}(\tau)$.

We emphasize that we are interested in those solutions of (3.12) which behave as in (3.10) as $|y| \to 0$. As a matter of fact, (3.10) plays the role of the matching condition between the inner and outer solutions.

To analyse equation (3.12), we make the following ansatz:

$$\Phi(y,\tau) \sim \varepsilon^{1/2}(\tau)Q(y), \qquad \tau \to +\infty. \tag{3.13}$$

If we plug this into (3.12) we obtain the following equation for Q:

$$\lambda Q = \Delta Q - \frac{y\cdot\nabla Q}{2} - \frac{Q}{4} + 4\pi\sqrt{3}\delta(y), \tag{3.14}$$

where:

$$\lambda = \frac{1}{2}\left(\lim_{\tau\to+\infty}\frac{\dot{\varepsilon}(\tau)}{\varepsilon(\tau)}\right) < 0.$$

The matching condition (3.10) becomes:

$$Q(y) = \frac{\sqrt{3}}{|y|} - \beta|y| + \dots, \qquad |y| \to 0, \tag{3.15}$$

where:

$$\beta = -\frac{\sqrt{3}}{8}\left(\lim_{\tau\to+\infty}\frac{\sigma(\tau)}{\varepsilon^2(\tau)}\right) = \frac{\sqrt{3}}{8}(1-4\lambda).$$

We need to complement problem (3.14), (3.15) with some growth condition on Q as $|y| \to +\infty$. Equation (3.14) admits two different asymptotics as $|y| \to +\infty$. Solutions of (3.14) have an algebraic growth in the first case, and they grow faster than exponentially in the second situation. This last behaviour is incompatible with the original function

$u(x,t)$ being bounded in the region $|x| \sim 1$, and for that reason it will be excluded. We then require:

$$Q(y) \;\text{ is algebraically bounded for }\; |y| \gg 1. \tag{3.16}$$

We thus end up with problem (3.14)-(3.16). We next show that this is an eigenvalue problem that can be solved only for a particular sequence of values of λ.

Taking into account (3.14) and the radial symmetry of Q we write:

$$Q'' + \left(\frac{2}{y} - \frac{y}{2}\right) Q' - \left(\lambda + \frac{1}{4}\right) = 0, \qquad y > 0.$$

In view of (3.15) we change variables by setting:

$$Q(y) = \frac{W(y)}{y}.$$

We then obtain the following equation for W:

$$W'' - \frac{1}{2} y W' + \left(\frac{1}{4} - \lambda\right) W = 0, \qquad y > 0.$$

If we make the further change of variables $z = y/2$ we derive:

$$W'' - 2zW' + (1 - 4\lambda) W = 0, \qquad y > 0, \tag{3.17}$$

which is the standard Hermite equation (cf. Szego 1978, p. 106). We then conclude that W (and therefore Q) has algebraic growth if and only if:

$$1 - 4\lambda = 2n \quad \Rightarrow \quad \lambda = \frac{1}{4} - \frac{n}{2}, \qquad n = 0, 1, 2, \ldots. \tag{3.18}$$

For these values of λ the corresponding solutions of (3.17) are given (up to a multiplicative constant) by the Hermite polynomials $H_n(z)$. Since however we are interested in radial solutions of (3.14), we need to exclude the odd values of n. In addition, we exclude the case $n = 0$ since it contradicts our basic assumption that $\varepsilon(\tau)$ decreases to zero (cf. the definition of λ in (3.14)). Thus, we retain the following solutions of (3.18):

$$\lambda = \frac{1}{4} - k, \qquad\qquad W_k(z) = C_k H_{2k}(z), \quad k = 1, 2, \ldots.$$

We still need to check the matching condition at the origin (3.15), which in our new variables is equivalent to:

$$W_k(z) = C_k H_{2k}(z) \sim \sqrt{3} - 4\frac{\sqrt{3}}{8}(1 - 4\lambda) z^2 = \sqrt{3}(1 - 2kz^2), \qquad z \to 0. \tag{3.19}$$

If we compute the first two terms of $H_{2k}(z)$ (cf. Appendix A) we see that:

$$H_{2k}(z) = \frac{(-1)^k (2k)!}{k!}(1 - 2kz^2 + \ldots),$$

in complete agreement with (3.19) if we choose:

$$C_k = \frac{(-1)^k k! \sqrt{3}}{(2k)!}, \qquad k = 1, 2, \ldots. \tag{3.20}$$

Returning to our original variables, we see that we have found a sequence of λ_k's to each of which there corresponds a solution of the problem (3.14)-(3.16), namely:

$$\lambda_k = \frac{1}{4} - k, \qquad Q_k(y) = C_k \frac{H_{2k}(\frac{y}{2})}{y}, \qquad k = 1, 2, \ldots. \tag{3.21}$$

From the definition of λ we have that $\dot{\varepsilon}(\tau) \approx 2\lambda_k \varepsilon(\tau)$ for $\tau \gg 1$, and consequently:

$$\varepsilon(\tau) \sim A e^{\left(\frac{1}{2} - 2k\right)\tau}, \qquad k = 1, 2, \ldots, \qquad A > 0, \qquad \tau \gg 1. \tag{3.22}$$

It is easy to check that the $\varepsilon(\tau)$ just found satisfy all our previous assumptions.

Once $\varepsilon(\tau)$ has been calculated we can obtain a detailed description of the asymptotics of $u(x,t)$ near the origin. Thus, for $|x| \sim A(T-t)^{2k}$ (or, equivalently $|y| \sim \varepsilon(\tau)$) we compute as $t \to T^-$:

$$\begin{aligned} u(x,t) &= (T-t)^{-1/4}\Phi(y,\tau) = (T-t)^{-1/4}\varepsilon^{-1/2}(\tau)G(\xi,\tau) \\ &\sim A^{-1/2} e^{k\tau} \overline{u}(\xi) \sim A^{-1/2}(T-t)^{-k}\overline{u}\left(\frac{x}{A(T-t)^{2k}}\right). \end{aligned} \tag{3.23}$$

In particular we have:

$$u(0,t) \sim \frac{1}{\sqrt{A}}(T-t)^{-k}, \qquad k = 1, 2, \ldots, \qquad t \to T^-.$$

Note that this estimate actually shows the occurrence of fast blow up. We next derive the final time profile of u:

$$u(x,t) = (T-t)^{-1/4}\Phi(y,\tau) = e^{\tau/4}\varepsilon^{1/2}(\tau)Q_k(y) \sim \sqrt{A} e^{\left(\frac{1}{2} - k\right)\tau} Q_k(x e^{\frac{\tau}{2}}).$$

Using the fact that:

$$Q_k(y) = C_k \frac{H_{2k}(\frac{y}{2})}{y} \sim C_k |y|^{2k-1}, \qquad y \to +\infty,$$

and taking the limit as $\tau \to +\infty$ we eventually obtain:

$$u(x,T) \sim C_k \sqrt{A}\, |x|^{2k-1}, \qquad |x| \to 0, \tag{3.24}$$

with C_k as in (3.20). It is interesting to notice that the function $u(x,T)$ is not singular at the origin. On the contrary, it has a zero at $x = 0$. Moreover, if $k \geq 2$ the function $u(x,T)$ is even differentiable at $x = 0$. This is a key characteristic feature of this blowup mechanism: the region where u grows unbounded becomes more and more concentrated near the origin, and in the limit as $t \to T^-$ this region disappears. In fact, using our previous results we can obtain more precise asymptotics of $u(x,t)$ as $t \to T^-$. Indeed, from (3.23) we easily see that:

$$\int_{I\!R^3} u^6(x,t)d^3x \approx \int_{I\!R^3} \bar{u}^6(\xi)d^3\xi = \frac{3\sqrt{3}\pi^2}{4}.$$

Taking also into account (3.24), the following (weak) convergence result follows:

$$u^6(x,t) \rightharpoonup K_1\delta(x) + K_2|x|^{6(2k-1)}, \qquad \text{as } \; t \to T^-,$$

where:

$$K_1 = \frac{3\sqrt{3}\pi^2}{4}, \qquad K_2 = C_k^6 A^3.$$

We note that because of (3.13), (3.21) and standard properties of the Hermite polynomials, the solutions just constructed change sign at least once for $k \geq 1$. Some indicative drawings of $u(x,t)$ near blow up for $k = 1, 2, 3$ are given in Figures 1, 2, 3 respectively in Appendix C.

We conclude this Section with a brief remark on stability. We expect the solutions just discussed to be unstable. At the formal level, this can be justified as follows. To begin with, it is readily seen from (3.18) (with $n = 0$ there) that (3.12) has a solution which grows exponentially in τ. Such a type of instability can be dispensed with by means of a change in the corresponding blow up time (actually, this is the rationale behind our former assumption $n \geq 1$ in (3.18)). However, the steady state equation that describes the inner region turns out to be irremediably unstable. For instance, this can be seen from the fact that, as $\tau \to +\infty$, equation (3.7) is dominated by:

$$\varepsilon^2\phi_\tau = \Delta\phi + p\bar{u}^{p-1}\phi = \mathcal{L}\phi.$$

The corresponding stationary equation $\mathcal{L}\phi = 0$ has at least one positive eigenvalue, since the operator $\mathcal{L}$ has a zero eigenvalue whose eigenfunctions change sign (these last are given by $\frac{\partial \bar{u}_\lambda}{\partial \lambda}$, where $\bar{u}_\lambda$ is the rescaled function defined right after (2.3)). This instability cannot be avoided by a shift of the blow up point, because of the radiality of the problem under consideration.

Note that we have described in detail the behaviour of our blowing up solutions locally around the origin. To actually obtain these solutions, one should start from initial values $u_0(r)$ which are close to the expected singularity profile near $r = 0$. The behaviour of $u_0(r)$ for $r \gg 1$ is largely irrelevant. For instance requiring $u_0(r) \to 0$ as $r \to +\infty$, algebraically, will do. The reader is referred to Herrero & Velázquez 1996 for details of a similar argument.

4 Space dimension $N = 4$

The basic strategy remains the same as in the previous Section. As before, we use in the inner region the approximation:

$$G(\xi,\tau) = \bar{u}(\xi) + \sigma(\tau)H(\xi) + \ldots.$$

However, a major difference with the three dimensional case stems from the fact that the asymptotics of the function $H(\xi)$ turns out to be (cf. Appendix B):

$$H(\xi) = 2\log\xi - \left(\frac{10}{3} + \log 8\right) + O\left(\frac{\log^2\xi}{\xi^2}\right), \qquad \xi \to +\infty. \tag{4.1}$$

Using the fact that $\Phi(y,\tau) = \varepsilon^{-1}G(\xi,\tau)$, the asymptotics of $\bar{u}$ and (4.1), we can write the analogue of (3.10):

$$\Phi(y,\tau) \sim \frac{8\varepsilon(\tau)}{|y|^2} - \frac{2\sigma(\tau)\log\varepsilon(\tau)}{\varepsilon(\tau)} + \frac{2\sigma(\tau)\log y}{\varepsilon(\tau)} - \frac{\sigma(\tau)}{\varepsilon(\tau)}\left(\frac{10}{3} + \log 8\right) + \ldots, \tag{4.2}$$

for $\tau \gg 1$. As in the case $N = 3$ we deduce that $\Phi \to 0$ in the region $|y| \sim 1$. There is however a major difference with the three dimensional case, since it is clear from (4.2) that the leading term in the region $|y| \sim 1$ is $-\frac{2\sigma(\tau)\log\varepsilon(\tau)}{\varepsilon(\tau)}$. For this reason the analysis in this case is rather different from the one performed in the previous Section. In particular we do not have to consider an eigenvalue problem analogous to (3.14)-(3.16). Instead, we will use the spectral properties of the operator A discussed in Section 2.

As we have previously done, we approximate the Φ-equation in the outer region (that is when $|y| \gg \varepsilon(\tau)$), by the following linear equation:

$$\Phi_\tau = \Delta\Phi - \frac{y\cdot\nabla\Phi}{2} - \frac{\Phi}{2} + 32\pi^2\varepsilon(\tau)\delta(y) = A\phi + 32\pi^2\varepsilon(\tau)\delta(y), \qquad \tau \gg 1. \tag{4.3}$$

Notice that the strength of the Dirac mass in (4.3) is given by (3.11) since we just use the same argument. We want to obtain solutions of (4.3) that match with the inner solution $G(\xi,\tau)$, that is, solutions that near the origin behave as in (4.2). To this end we write:

$$\Phi(y,\tau) = a(\tau)\phi_l(y) + \Omega(y,\tau), \qquad <\phi_l,\,\Omega> = 0, \tag{4.4}$$

where $\phi_l(y)$, $l = 1, 2, \ldots$, is one of the eigenfunctions of the linear operator A described in Section 2, and the inner product $<\cdot>$ has been defined in (2.2). The idea behind the decomposition (4.4) is the following: we may formally expand the solution of (4.3) in its Fourier modes as:

$$\Phi(y,\tau) = \sum_{l=0}^{\infty} a_l(\tau)\phi_l(y).$$

We now make the assumption that the large time behavior of Φ is driven by one of the eigenvalues of A, say the l-eigenvalue. By distinguishing then the l eigenfunction from the rest of the terms we arrive at (4.4). We then expect to obtain a matching of the term $a(\tau)\phi_l(y)$ with the term $\frac{2\sigma(\tau)\log\varepsilon(\tau)}{\varepsilon(\tau)}$.

Using (4.3) and (4.4), we easily obtain the following equations for $a(\tau)$ and $\Omega(y,\tau)$ respectively:

$$\dot{a}(\tau) = -(l + \frac{1}{2})a + 32\pi^2\varepsilon(\tau) <\delta,\phi_l> . \tag{4.5}$$

$$\Omega_\tau = \Delta\Omega - \frac{y\cdot\nabla\Omega}{2} - \frac{1}{2}\Omega + 32\pi^2\varepsilon(\tau)(\delta(y) - \phi_l(y) < \delta, \phi_l >). \tag{4.6}$$

Taking into account the definition of $\sigma(\tau)$, we see that the term $\frac{2\sigma(\tau)\log\varepsilon(\tau)}{\varepsilon(\tau)}$ is of order $\varepsilon\log\varepsilon$. On the other hand, the term $a(\tau)\phi_l(y)$ is driven by the l-eigenvalue of A and therefore behaves like $e^{-(l+\frac{1}{2})\tau}$ with some algebraic corrections (cf. (4.5)). Consequently, if the terms $a(\tau)\phi_l(y)$ and $\frac{2\sigma(\tau)\log\varepsilon(\tau)}{\varepsilon(\tau)}$ are to match, we should expect that ε behaves roughly as $e^{-(l+\frac{1}{2})\tau}$ up to some algebraic factors. We then define for convenience:

$$\varepsilon(\tau) = e^{-(l+\frac{1}{2})\tau}g(\tau), \tag{4.7}$$

where $g(\tau)$ is expected to behave algebraically. In addition we assume that $|\dot{g}(\tau)| \ll |g(\tau)|$.

We now compute the asymptotics of Ω. Making the ansatz:

$$\Omega(y,\tau) \sim \varepsilon(\tau)Q(y), \qquad \tau \to +\infty,$$

and using (4.7), we find that Q satisfies for $\tau \gg 1$:

$$-(l+\frac{1}{2})Q = \Delta Q - \frac{y\cdot\nabla Q}{2} - \frac{1}{2}Q + 32\pi^2(\delta(y) - \phi_l(y) < \delta, \phi_l >). \tag{4.8}$$

By Fredholm's theory, equation (4.8) has a unique solution Q (algebraically bounded as $|y| \to +\infty$) satisfying $< Q, \phi_l >= 0$. Since $< \delta, \phi_l >= \phi_l(0)$, the source term in (4.8) is equal to $32\pi^2\delta(y) - \Gamma\phi_l(y)$, with:

$$\Gamma = 32\pi^2\phi_l(0). \tag{4.9}$$

We can then compute the asymptotics of Q near $y = 0$. (We can even write down the solution of (4.8)). We finally obtain:

$$Q(y) \sim \frac{8}{y^2} - 4(l+1)\log y + D_l + O(1), \qquad y \to 0, \tag{4.10}$$

where D_l is a constant uniquely determined by the orthogonality condition $< Q, \phi_l > = 0$. Returning now to the Φ function, and taking into account (4.4) and our previous calculations, we end up with the following behaviour for $\tau \gg 1$:

$$\Phi(y,\tau) \sim \frac{8\varepsilon(\tau)}{y^2} + a(\tau)\phi_l(0) - 4\varepsilon(\tau)(l+1)\log y + D_l\varepsilon(\tau) + \ldots, \qquad y \to 0. \tag{4.11}$$

Using (4.7) and the definition of σ we easily see that $\frac{2\sigma(\tau)}{\varepsilon(\tau)} = 4\varepsilon(\tau)(l+1)$ as $\tau \to +\infty$. Comparing then (4.11) and (4.2) we see that the first and third terms match. To obtain then a complete matching we require:

$$-\frac{2\sigma(\tau)\log\varepsilon(\tau)}{\varepsilon(\tau)} - \frac{\sigma(\tau)}{\varepsilon(\tau)}\left(\frac{10}{3} + \log 8\right) = a(\tau)\phi_l(0) + D_l\varepsilon(\tau). \tag{4.12}$$

This equation should be complemented with the equation (4.5) satisfied by $a(\tau)$. If we put:

$$a(\tau) = e^{-(l+\frac{1}{2})\tau} h(\tau), \tag{4.13}$$

and take into account (4.7), we can write system (4.5), (4.12) as:

$$\dot{h}(\tau) = \Gamma g(\tau), \tag{4.14}$$

$$\left(-(1+2l)\tau + 2\log g + \frac{10}{3} + \log 8\right)(\dot{g} - 2(l+1)g) = -(\phi_l(0)h + D_l g), \tag{4.15}$$

with Γ as in (4.9). Recalling our assumption $|\dot{g}| \ll |g|$ as $\tau \to +\infty$, and keeping the most important terms in (4.15), we obtain:

$$g(\tau) = -\frac{\phi_l(0)}{2(1+2l)(1+l)\tau} h(\tau).$$

If we plug in this in (4.14) we obtain an ODE for h that we can solve, so that eventually we deduce:

$$h(\tau) \sim \frac{A}{\tau^{\nu_l}}, \qquad \tau \to +\infty, \tag{4.16}$$

where A is an arbitrary constant, and:

$$\nu_l = \frac{16\pi^2 \phi_l^2(0)}{(1+2l)(1+l)} = \frac{1}{1+2l}, \tag{4.17}$$

where in the last equality we used the fact that $\phi_l^2(0) = \frac{l+1}{16\pi^2}$ for $N=4$; cf. Appendix A. We then have that:

$$g(\tau) \sim \frac{A}{(2l+1)(2l+2)} \cdot \frac{1}{\tau^{\nu_l+1}},$$

and finally, taking into account (4.7) and (4.16):

$$\varepsilon(\tau) \sim \frac{A}{(2l+1)(2l+2)}\, e^{-(l+\frac{1}{2})\tau}\, \tau^{-\frac{2l+2}{2l+1}}, \qquad \tau \to +\infty. \tag{4.18}$$

We can then go back to the original variables $u(x,t)$ and compute near the origin ($|y| \sim \varepsilon(\tau)$):

$$u(x,t) = (T-t)^{-1/2}\Phi(y,\tau) \sim (T-t)^{-1/2}\varepsilon^{-1}(\tau)\bar{u}(\xi), \qquad \tau \to +\infty.$$

In particular using our previous results we arrive at:

$$u(0,t) \sim \frac{(2l+1)(2l+2)}{A}\,(T-t)^{-(l+1)}\,|\log(T-t)|^{\frac{2l+2}{2l+1}}, \qquad l = 0,1,\ldots, \qquad t \to T^-.$$

We next derive the final time profile. In the region $|y| \gg 1$, $|y|$ fixed, $\tau \to +\infty$, we have the asymptotics:

$$\Phi(y,\tau) \sim a(\tau)\phi_l(y) \sim -\frac{A\,(-1)^l\, C_{l,4}\, e^{-(l+\frac{1}{2})\tau}}{\tau^{\nu_l}\, l!}\left(\frac{y^2}{4}\right)^l,$$

where we used (4.13), (4.16) and the fact that $\phi_l(z) \sim \frac{C_{l,4}(-1)^l}{l!} z^l$ as $z \to +\infty$; cf. Appendix A. We now argue as in Herrero & Velázquez (1997). Given x close to zero, let us take $\bar{t}$ such that $|x| = B(T-\bar{t})^{1/2}$, where B is a large enough and fixed number. We then obtain the approximation:

$$u(x,\bar{t}) \sim -\frac{A\,(-1)^l\,C_{l,4}}{|\log(T-\bar{t})|^{\nu_l}}\,\frac{|x|^{2l}}{4^l\,l!},$$

which is valid in regions $|x| \sim B(T-\bar{t})^{1/2}$ with $B>0$ fixed (but large). Since $u(x,t)$ solves (1.1), and only experiences changes at distances $O((T-\bar{t})^{1/2})$ in times of order $O((T-\bar{t}))$, u remains basically constant in these regions for subsequent times, and therefore:

$$u(x,T) \sim -\frac{A\,(-1)^l\,C_{l,4}}{|\log(\frac{|x|^2}{B})|^{\nu_l}}\,\frac{|x|^{2l}}{4^l\,l!}, \qquad |x| \ll 1.$$

Taking the limit as $|x| \to 0$, we can neglect the constant B, and thus deduce the final profile:

$$u(x,T) \sim -\frac{A\,(-1)^l\,C_{l,4}}{|\log(|x|^2)|^{\nu_l}}\,\frac{|x|^{2l}}{4^l\,l!}, \qquad |x| \to 0,$$

or, if we use the values of $C_{l,4} = \frac{1}{4\pi\sqrt{l+1}}$ and ν_l (see (4.17)):

$$u(x,T) \sim \frac{A(-1)^{l+1}}{2^{\frac{1}{2l+1}}\,4^{l+1}\,l!\,\pi\,\sqrt{l+1}}\,\frac{|x|^{2l}}{|\log|x||^{\frac{1}{2l+1}}}, \qquad |x| \to 0.$$

If $l \geq 1$ the profiles just obtained are essentially the same as in the case $N=3$, except for the presence of a logarithmic correction. However the case $l=0$ is different, since it yields a peaked and negative profile at time $t=T$. This behaviour is shown in Figure 4, Appendix C.

Finally, we remark that, arguing as in the case $N=3$, one readily sees that the solutions just obtained are expected to be unstable when $l \geq 1$. Such arguments, however, do not apply when $l=0$, and this case remains undecided from the point of view of the stability. A similar situation occurs for the cases $N \geq 5$ to be discussed below.

5 Space dimension $N \geq 5$

For dimensions $N \geq 5$ the asymptotics of $H(\xi)$ is different from before. Namely (cf. Appendix B):

$$H(\xi) = \frac{N(N-2)B_N}{4} + O(\frac{1}{\xi}), \qquad \xi \to +\infty, \tag{5.1}$$

where B_N is a positive constant given in (B.4). We then obtain the following inner matching condition for $\varepsilon \ll |y| \ll 1$:

$$\Phi(y,\tau) \sim \frac{N(N-2)B_N}{4}\,\frac{2\varepsilon\dot{\varepsilon}-\varepsilon^2}{\varepsilon^{\frac{N-2}{2}}} + (N(N-2))^{\frac{N-2}{2}}\,\frac{\varepsilon^{\frac{N-2}{2}}}{|y|^{N-2}} + \dots \qquad \tau \to +\infty. \tag{5.2}$$

Notice that the leading term now comes from the constant term of $H(\xi)$.

In the region $|y| \sim 1$ we can approximate the Φ-equation (2.1) by:

$$\Phi_\tau = \Delta\Phi - \frac{y\cdot\nabla\Phi}{2} - \frac{\Phi}{p-1} + c(\tau)\delta(y), \qquad \tau \to +\infty, \tag{5.3}$$

where $c(\tau)$ is given by (3.11).

In view of (5.2) we easily see that in the region $|y| \sim 1$, the contribution of the Dirac mass in (5.3) is negligible to the lowest order. We then expect that the most important contribution in this region will come from the homogeneous part. As in the case $N = 4$, that will be driven by some eigenvalue of the operator A, say the l-eigenvalue. In conclusion, to the leading order, the asymptotics of Φ in the outer region is given by:

$$\Phi(y,\tau) \sim -Ae^{-(l+\frac{1}{p-1})\tau}\phi_l(y) + \dots, \qquad \tau \to +\infty, \qquad A > 0. \tag{5.4}$$

The minus sign has been chosen in order to be in agreement with (5.2) (notice that $\phi_l(0) > 0$). Equating then the leading terms of (5.2) and (5.4), we obtain the following matching condition:

$$\frac{N(N-2)B_N}{4}\,\frac{2\varepsilon\dot{\varepsilon} - \varepsilon^2}{\varepsilon^{\frac{N-2}{2}}} = -Ae^{-(l+\frac{1}{p-1})\tau}\phi_l(0), \qquad A > 0. \tag{5.5}$$

Case $N = 5$

In the special case $N = 5$, $p = 7/3$, we can write (5.5) as:

$$\frac{d}{d\tau}(\varepsilon^{1/2}) - \frac{1}{4}\varepsilon^{1/2} = -\frac{A\phi_l(0)}{15B_5}\,e^{-(l+\frac{3}{4})\tau},$$

whence:

$$\varepsilon(\tau) \sim Ke^{-(2l+\frac{3}{2})\tau}, \qquad \tau \to +\infty,$$

with:

$$K = \frac{A^2\phi_l^2(0)}{15^2 B_5^2 (l+1)^2}.$$

Arguing as in the previous cases, we now obtain:

$$u(0,t) \sim K^{-3/2}(T-t)^{-(l+1)}, \qquad l = 0, 1, \dots, \qquad t \to T^-.$$

The final time profile is given by:

$$u(x,T) \sim -\frac{AC_{l,N}(-1)^l}{4^l\, l!}\,|x|^{2l}, \qquad |x| \to 0.$$

Of particular interest is the case $l = 0$, since in the limit, $u(x,T)$ becomes flat and negative, see Figure 5, Appendix C.

Case $N = 6$

In the special case $N = 6$, $p = 2$, if we try to satisfy the matching condition (5.5), we see that the left hand side is of order one whereas the right hand side decays exponentially fast for $l = 0, 1, 2, \ldots$. Therefore such a matching is impossible. Let us rewrite (5.2) as follows:

$$\Phi(y,\tau) \sim 6B_6\left(2\frac{\dot{\varepsilon}}{\varepsilon} - 1\right) + 24^2\frac{\varepsilon^2}{|y|^4} + \ldots, \qquad \tau \to +\infty. \tag{5.6}$$

This expansion suggests that in order to obtain a matching, we should look in the outer region $|y| \sim 1$, for a solution of the Φ equation that is of order one (and of negative sign). We then take as outer solution the constant $-(p-1)^{-\frac{1}{p-1}} = -1$, which is an exact solution of the Φ-equation (2.1). The following matching condition follows:

$$6B_6\left(2\frac{\dot{\varepsilon}}{\varepsilon} - 1\right) = -1 \qquad \Rightarrow \qquad \varepsilon(\tau) = Ae^{-\left(\frac{1}{12B_6} - \frac{1}{2}\right)\tau}. \tag{5.7}$$

At this point it is important to know the exact value of the constant B_6. Using formula (B.4) we calculate that $B_6 = 1/15$. Returning now to (5.7) we conclude that:

$$\varepsilon(\tau) \sim Ae^{-\frac{3}{4}\tau}, \qquad A > 0, \qquad \tau \gg 1.$$

As a matter of fact we can obtain a more precise approximation of $\varepsilon(\tau)$ (including correction terms) in the following way. Let us recall that in the subcritical case it is well known that there exist positive and radially symmetric solutions of the Φ-equation, approaching the constant $(p-1)^{-\frac{1}{p-1}}$, and for which the following asymptotics hold (cf. Velázquez 1992 for the exact statement):

$$\Phi(y,\tau) \sim (p-1)^{-\frac{1}{p-1}} + \frac{(p-1)^{-\frac{1}{p-1}}}{2p\tau}\left(n - \frac{1}{2}|y|^2\right) + o(\frac{1}{\tau}) = 1 + \frac{1}{4\tau}\left(6 - \frac{1}{2}|y|^2\right) + o(\frac{1}{\tau}).$$

If we then assume that such solutions (with negative sign) continue to exist in our case as well, we end up with the following matching condition:

$$6B_6\left(2\frac{\dot{\varepsilon}}{\varepsilon} - 1\right) = -1 - \frac{3}{2\tau} + o(\frac{1}{\tau}),$$

from which we obtain:

$$\varepsilon(\tau) \sim Ae^{-\frac{3}{4}\tau}\tau^{-\frac{15}{8}}, \qquad A > 0, \qquad \tau \gg 1.$$

It then follows that:

$$u(0,t) \sim A^{-1}(T-t)^{-1/4}|\ln(T-t)|^{-15/8}, \qquad t \to T^-,$$

whereas the final time profile is expected to be the same as in the subcritical case (see Velázquez 1992):

$$u(x,T) \sim -\frac{16|\ln|x||}{|x|^2}, \qquad |x| \to 0.$$

Case $N \geq 7$

In this case the term $(2\varepsilon\dot{\varepsilon} - \varepsilon^2)\varepsilon^{-\frac{N-2}{2}}$ (cf. (5.2)) grows with τ. It is not then clear what an appropriate outer solution might be, that could match with this term. It seems that new ideas are required in order to understand this situation.

6 Positive and radially decreasing solutions

In this Section we consider blowing up solutions of equation (1.4) that are positive and radially decreasing. We show that under these assumptions there exists no fast blow up mechanism. That is, the blow-up rate is the self-similar one:

$$u(0,t) \leq C(T-t)^{-\frac{1}{p-1}}, \qquad C > 0, \qquad 0 < t < T. \tag{6.1}$$

This is done by comparison with suitable self-similar solutions of (1.1) having the same blowup time T.

To proceed, we set:

$$y = x(T-t)^{-1/2}, \qquad u(x,t) = (T-t)^{-\frac{1}{p-1}}\Phi(y),$$

so that Φ satisfies the equation:

$$\Delta\Phi - \frac{y \cdot \nabla\Phi}{2} - \frac{\Phi}{p-1} + \Phi^p = 0. \tag{6.2}$$

Let us denote by Φ_m the solution of (6.2) satisfying:

$$\Phi_m(0) = m > 0, \qquad \Phi_m'(0) = 0.$$

We claim that $\Phi_m(y)$ changes sign at a point y_m as $m \to +\infty$. To see why this happens let us present a formal argument. To begin with, we change variables by:

$$\xi = m^{\frac{p-1}{2}} y, \qquad \Phi_m(y) = mG(y),$$

so that G satisfies:

$$\Delta G - m^{-(p-1)}\left(\frac{\xi \cdot \nabla G}{2} + \frac{G}{p-1}\right) + G^p = 0, \qquad G(0) = 1, \quad G'(0) = 0. \tag{6.3}$$

As $m \to +\infty$ we expect that G will approach $\bar{u}$. To obtain a better approximation we write:

$$G(\xi) = \bar{u}(\xi) - m^{-(p-1)}H(\xi). \tag{6.4}$$

Plugging this into the G-equation we see that H satisfies to the lowest order:

$$\Delta H + p\bar{u}^{p-1}H + \bar{\phi}_0 = 0, \qquad H(0) = H'(0) = 0,$$

with $\bar{\phi}_0$ as in (2.5). Thus, H denotes the same function as in the previous Sections (cf. (3.8), (4.1),..).

Since the asymptotic behaviour of H depends on the dimension N, let us consider first the case $N = 3$. Taking into account (6.4), as well as the asymptotics of $\bar{u}$ and H in the case where $N = 3$, we readily see that:

$$G(\xi) \sim \frac{\sqrt{3}}{\xi} - m^{-4}\frac{\sqrt{3}\,\xi}{8}, \qquad m, \xi \to +\infty.$$

Consequently, there exists a point $\xi_m \sim m^2$ at which G crosses the ξ -axis. Returning to the $y = m^{-2}\xi$ variable, we see that $\Phi_m(y)$ crosses the y-axis at a point $y_m \sim O(1)$ as $m \to +\infty$. If we repeat the same argument for $N = 4$ and $N \geq 5$ we see that $G(\xi)$ and $\Phi_m(y)$ change sign at ξ_m and y_m respectively, where as $m \to +\infty$:

$$\begin{aligned}
&\xi_m = O(m^2), && y_m = O(1), && N = 3,\\
&\xi_m = O(\frac{m}{\sqrt{\log m}}), && y_m = O(\frac{1}{\sqrt{\log m}}), && N = 4, \qquad (6.5)\\
&\xi_m = O\left(m^{\frac{(p-1)^2}{4}}\right), && y_m = O\left(m^{\frac{(p-1)(p-3)}{4}}\right), && N \geq 5.
\end{aligned}$$

Notice that when $N \geq 5$ we always have that $(p-1)(p-3) < 0$. In conclusion we have that in all cases $\Phi_m(y)$ crosses the y-axis at a point y_m, such that $y_m < C$ as $m \to +\infty$, when $N = 3$, and $y_m \to 0$ as $m \to +\infty$, for $N \geq 4$. Consequently, as m increases, the function $\Phi_m(y)$ decays very rapidly from the value $\Phi_m(0) = m$ to $\Phi_m(y_m) = 0$.

Returning to the original variables, we have obtained a radially symmetric solution of (1.1), namely:

$$u_m(x,t) = (T-t)^{-\frac{1}{p-1}}\Phi_m(y),$$

such that:

$$u_m(0,t) = m(T-t)^{-\frac{1}{p-1}}, \qquad (6.6)$$

and $u_m(0,t)$ crosses the x-axis at $\pm x_m$, with $x_m = y_m(T-t)^{1/2}$ and y_m as in (6.5).

We are now ready to implement the comparison argument. Let $u(x,t)$ be a positive, radially symmetric and radially decreasing solution of (1.1), which blows up at time T. By choosing m sufficiently large, we can ensure that $u_m(|x|,t_0)$ (as defined before) and $u(|x|,t_0)$ have exactly two points of intersection in the interval $-x_m < |x| < x_m$, at some fixed time $0 \leq t_0 < T$. It then follows by an intersection comparison argument (cf. for instance, Angenent 1988, Galaktionov & Posashkov 1986) that $u_m(x,t)$ and $u(x,t)$ will keep having two points of intersection in $(-x_m, x_m)$ for all later times $t_0 \leq t < T$. In particular it then turns out that:

$$u(0,t) < u_m(0,t) = m(T-t)^{-\frac{1}{p-1}}, \qquad t < T,$$

which is the upper bound we are looking for.

It is not difficult to make the above result rigorous. As a matter of fact all we need to prove is that the solution $G(\xi, m)$ of (6.3) does indeed cross the ξ-axis at a point ξ_m as given in (6.5). We do this in the following Lemma.

Lemma 6.1 *Let $G(\xi, m)$ be the solution of (6.3). Then, for m large enough, there exists a point ξ_m such that $G(\xi_m, m) = 0$ and estimates (6.5) hold.*

Proof: We will give the proof in the case $N = 3$, $p = 5$, the other cases being quite similar. We set:

$$\phi(\xi, m) = G(\xi, m) - \bar{u}(x) + m^{-4}H(\xi), \tag{6.7}$$

with $\bar{u}$ and H as defined before. We will prove that given any positive constant C we have that:

$$|\phi(\xi, m)| = o(m^{-4}), \qquad \text{as } m \to +\infty, \quad \text{uniformly for} \quad |\xi| \leq Cm^2. \tag{6.8}$$

The desired result then follows from (6.7) and (6.8) if we take into account the asymptotics of $\bar{u}$ and H. To this end we define:

$$\tilde{\xi}_m = \sup_{0 \leq \xi < +\infty} \{\xi : \ |\phi(\xi, m)| + |\xi \, \phi_\xi(\xi, m)| < \delta m^{-4}\}, \tag{6.9}$$

where δ is a small positive fixed number. We will show that given any constant $C > 0$, we have that:

$$\tilde{\xi}_m > Cm^2, \qquad\qquad m \to +\infty. \tag{6.10}$$

Estimate (6.8) then follows at once from (6.9) and (6.10). To prove (6.10) we will argue by contradiction. More precisely, assuming in the rest of the proof that:

$$\tilde{\xi}_m \leq Cm^2, \tag{6.11}$$

for some positive constant C, we will contradict the maximality of $\tilde{\xi}_m$, defined in (6.9).

To proceed further, we observe that plugging (6.7) in (6.3), and taking into account the equations satisfied by $\bar{u}$ and H, there holds:

$$\begin{aligned} &\Delta\phi + \Big((\bar{u} + \phi - m^{-4}H)^5 - \bar{u}^5 + 5m^{-4}\bar{u}^4 H\Big) \\ &= m^{-4}\left(\frac{\xi\phi_\xi}{2} + \frac{\phi}{4} - m^{-4}\frac{\xi H_\xi}{2} - m^{-4}\frac{H}{4}\right), \qquad \phi(0) = \phi'(0) = 0. \end{aligned} \tag{6.12}$$

Using (6.9) and the properties of H we then obtain that, as $m \to +\infty$:

$$\Delta\phi + Q(\xi, m) = O(m^{-8}), \qquad 0 \leq \xi \leq \tilde{\xi}_m \leq Cm^2, \tag{6.13}$$

with:

$$Q(\xi, m) = (\bar{u} + \phi - m^{-4}H)^5 - \bar{u}^5 + 5m^{-4}\bar{u}^4 H.$$

We next estimate Q. If $0 \leq \xi \leq R$, for some constant R large enough, we have that:

$$Q(\xi, m) = 5\bar{u}^4\phi + O(m^{-8}), \qquad\qquad m \to +\infty.$$

On the other hand, if $R < \xi \leq \tilde{\xi}_m \leq Cm^2$, using the asymptotics of $\bar{u}$ we deduce that:

$$Q(\xi, m) = 5\bar{u}^4\phi + O(m^{-8}\xi^{-1}), \qquad\qquad m, \ \xi \to +\infty.$$

Thus, for $0 < \xi \le \tilde{\xi}_m \le Cm^2$ we have:

$$\Delta\phi = 5\bar{u}^4\phi + F(\xi, m), \qquad \phi(0) = \phi'(0) = 0, \tag{6.14}$$

with:

$$F(\xi, m) \le \frac{Cm^{-8}}{1+\xi}, \qquad m \to +\infty.$$

Equation (6.14) can be integrated to yield:

$$\phi(\xi, m) = \bar{\phi}_0(\xi) \int_0^{\xi} \frac{d\eta}{\eta^2 \bar{\phi}_0^2(\eta)} \int_0^{\eta} \lambda^2 \bar{\phi}_0(\lambda) F(\lambda, m) d\lambda.$$

Using the estimates on F, we then obtain:

$$|\phi(\xi, m)| + |\xi\, \phi_{\xi}(\xi, m)| \le \frac{Cm^{-8}}{1+\xi}, \qquad \text{when } m \to +\infty.$$

Recalling our assumption $0 \le \xi \le \tilde{\xi}_m \le Cm^2$, we then deduce that for ξ in this range:

$$|\phi(\xi, m)| + |\xi\, \phi_{\xi}(\xi, m)| \le Cm^{-8}\xi \le Cm^{-6} \ll \delta m^{-4}, \qquad m \to +\infty,$$

which contradicts the maximality of $\tilde{\xi}_m$. This completes the proof of the Lemma.

7 Exponential nonlinearity

In this Section we consider the following semilinear heat equation with exponential nonlinearity in space dimension $N = 2$:

$$u_t = \Delta u + e^u, \qquad \text{in } \mathbb{R}^2. \tag{7.1}$$

We will show that solutions of (7.1) that are radially symmetric and decreasing exhibit no fast blow-up. That is, we will show that:

$$u(0, t) \le -\log(T-t) + C, \qquad C > 0. \tag{7.2}$$

The method we use is the same as in the previous Section, where we show the analogous result for critical power nonlinearities. We start by introducing similarity variables by:

$$y = x(T-t)^{-1/2}, \qquad u(x, t) = -\log(T-t) + \Phi(y),$$

so that Φ satisfies:

$$\Delta\Phi - \frac{y \cdot \nabla\Phi}{2} + e^{\Phi} - 1 = 0. \tag{7.3}$$

We then look for solutions Φ_ε of (7.3) satisfying:

$$\Phi_\varepsilon(0) = -2\log\varepsilon, \qquad \Phi_\varepsilon'(0) = 0,$$

for some small positive constant ε. Proceeding as before we introduce new variables by setting:

$$\xi = y/\varepsilon, \qquad \Phi = -2\log\varepsilon + G(\xi),$$

so that G satisfies the equation:

$$\Delta G + e^G - \varepsilon^2\left(\frac{\xi\cdot\nabla G}{2} + 1\right) = 0, \qquad G(0) = G'(0) = 0. \tag{7.4}$$

As ε tends to zero we (formally) have that G behaves like $\bar{u}$, where $\bar{u}$ now solves the equation:

$$\bar{u}'' + \frac{\bar{u}'}{\xi} + e^{\bar{u}} = 0, \qquad \bar{u}(0) = \bar{u}'(0) = 0.$$

The solution of this is then given by:

$$\bar{u}(\xi) = -2\log\left(\frac{\xi^2}{8} + 1\right).$$

To obtain a better approximation of G we set:

$$G(\xi) = \bar{u}(\xi) - \varepsilon^2 H(\xi),$$

so that H satisfies (to the lowest order):

$$\Delta H + e^{\bar{u}}H + \frac{\xi\cdot\nabla\bar{u}}{2} + 1 = 0, \qquad H(0) = H'(0) = 0. \tag{7.5}$$

Equation (7.5) can be studied as in the previous Sections. Then there holds:

$$H(\xi) \sim \frac{1}{4}\xi^2, \qquad \xi \to +\infty.$$

Summarizing we have that:

$$\Phi_\varepsilon(y) = -2\log\varepsilon + G(\xi) \sim -2\log\varepsilon - 2\log\left(\frac{\xi^2}{8} + 1\right) - \frac{\varepsilon^2}{4}\xi^2, \qquad \xi \to +\infty.$$

It then follows that $\Phi_\varepsilon(y)$ changes sign at a point $\xi_\varepsilon = O(\varepsilon^{-1/2})$, or equivalently, $y_m = O(\varepsilon^{1/2})$. The rest of the argument is the same as in the previous Section. A suitable comparison function is now defined by:

$$u_\varepsilon(x,t) = -\log(T-t) + \Phi_\varepsilon\left(\frac{x}{(T-t)^{1/2}}\right).$$

To make the proof rigorous we only need the analogue of Lemma 6.1:

Lemma 7.1 *Let $G(\xi,\varepsilon)$ be a solution of (7.4). Then, for $\varepsilon > 0$ small enough, there exists a point $\xi_\varepsilon = O(\varepsilon^{-1/2})$ such that:*

$$-2\log\varepsilon + G(\xi,\varepsilon) = 0.$$

The proof is quite similar to that of Lemma 6.1 and we therefore omit it.

Appendix

A. About Laguerre and Hermite polynomials

In order to derive the spectral properties of the operator A, defined in Section 2, let us study the following problem:

$$\Delta\phi - \frac{y \cdot \nabla\phi}{2} = \lambda\phi. \tag{A.1}$$

Since we are interested in radial functions only, we rewrite (A.1) as:

$$\phi'' + \left(\frac{N-1}{y} - \frac{y}{2}\right)\phi' - \lambda\phi = 0.$$

We next change variables by setting $x = y^2/4$ so that ϕ satisfies:

$$x\phi'' + \left(\frac{N}{2} - x\right)\phi' - \lambda\phi = 0.$$

This equation admits polynomial solutions if and only if $\lambda = -k$, $k = 0, 1, 2, \ldots$; cf. Szego (1978), p. 100. The corresponding solutions are given (up to a multiplicative constant) by $\phi_k(x) = L_k^{(\frac{N-2}{2})}(x)$, where $L_k^{(\frac{N-2}{2})}(x)$ denotes the modified Laguerre polynomials, which are given by:

$$L_k^{(a)}(x) = x^{-a}e^x \frac{d^k}{dx^k}\left(x^{a+k}e^{-x}\right), \qquad a > -1.$$

The $L_k^{(a)}$'s given above satisfy the orthogonality conditions:

$$\int_0^\infty e^{-x}x^a L_n^{(a)}(x)L_m^{(a)}(x)dx = \frac{\Gamma(a+n+1)}{n!}\delta_{n,m}, \tag{A.2}$$

where $\delta_{n,m} = 0$ if $n \neq m$ and $\delta_{n,n} = 1$, and:

$$L_k^{(a)}(0) = \left(\begin{array}{c} k+a \\ k \end{array}\right), \qquad L_k^{(a)}(x) \sim \frac{(-1)^k}{k!}x^k, \qquad x \to +\infty.$$

The radial eigenfunctions of A are then given by:

$$\phi_k(y) = C_{k,N} L_k^{(\frac{N-2}{2})}(\frac{y^2}{4}), \qquad k = 0, 1, \ldots.$$

We next select the normalization constants $C_{k,N}$ in order to have $\|\phi_k\| = 1$. Namely we impose:

$$\int_{I\!R^N} C_{k,N}^2 \left(L_k^{(\frac{N-2}{2})}\left(\frac{y^2}{4}\right)\right)^2 e^{-\frac{y^2}{4}} d^N y = 1.$$

Because of the radial symmetry we write:

$$d^N y = K_{N-1} y^{N-1} dy, \qquad\qquad K_{N-1} = \frac{2\pi^{N/2}}{\Gamma(N/2)}.$$

We next change variables by setting $x = y^2/4$, so that $y^{N-1}dy = 2(4x)^{\frac{N-2}{2}}dx$. Thus we have:

$$1 = 2K_{N-1}C_{k,N}^2 4^{\frac{N-2}{2}} \int_0^\infty e^{-x} x^{\frac{N-2}{2}} \left(L_n^{(\frac{N-2}{2})}(x)\right)^2 dx = 2^{N-1} K_{N-1} C_{k,N}^2 \frac{\Gamma(\frac{N}{2}+k)}{k!},$$

where in the last equality the orthogonality condition (A.2) has been taken into account. Hence we derive:

$$C_{k,N}^2 = \frac{\Gamma(\frac{N}{2})k!}{2^N \pi^{N/2} \Gamma(\frac{N}{2}+k)}. \tag{A.3}$$

It is also easy to compute $\phi_k(0)$:

$$\phi_k(0) = C_{k,N} L_n^{(\frac{N-2}{2})}(0) = C_{k,N} \begin{pmatrix} k + \frac{N-2}{2} \\ k \end{pmatrix}.$$

We next recall a few things about Hermite polynomials (cf. Szego 1978, p. 106). Consider the ODE:

$$W'' - 2xW' + 2nW = 0.$$

This equation admits algebraically bounded solutions if and only if $n = 0, 1, 2, \ldots$ and the corresponding solutions are (up to a multiplicative constant) the Hermite polynomials denoted by $H_n(x)$. They are given by the formula:

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2},$$

or alternatively,

$$\frac{H_n(x)}{n!} = \sum_{\nu=0}^{[\nu/2]} \frac{(-1)^\nu}{\nu!} \frac{(2x)^{n-2\nu}}{(n-2\nu)!}.$$

Using this second representation we compute the first few terms of $H_{2k}(x)$:

$$H_{2k}(x) = \frac{(-1)^k (2k)!}{k!} \left(1 - 2kx^2 + \frac{2^4 k(k-1)x^4}{4!} + \ldots\right). \tag{A.4}$$

Concerning its behavior at infinity we have that:

$$H_n(x) \sim (2x)^n, \qquad\qquad x \to +\infty.$$

We finally note for completeness that Hermite polynomials can be reduced to Laguerre polynomials with the parameter $a = \pm\frac{1}{2}$.

B. Asymptotics of $H(\xi)$

Here we will compute the asymptotics of $H(\xi)$ as $\xi \to +\infty$ for several values of the space dimension N. Let us recall that $H(\xi)$ satisfies equation (3.9), which because of the radial symmetry is written as:

$$H''(\xi) + \frac{N-1}{\xi} H'(\xi) + p\bar{u}^{p-1}(\xi) H(\xi) + \bar{\phi}_0(\xi) = 0, \qquad H(0) = H'(0) = 0.$$

with $\bar{\phi}_0$ defined by (2.5). This equation is a linear nonhomogeneous ODE of which we know a solution of the corresponding homogeneous part, namely $\bar{\phi}_0(\xi)$ (cf Section 2). Consequently we look for H in the form $H = \bar{\phi}_0 f$. Plugging this into the H-equation we obtain after some straightforward calculations that f is given by:

$$H = \bar{\phi}_0 f, \qquad f(\xi) = -\int_0^{\xi} \frac{d\eta}{\eta^{N-1} \bar{\phi}_0^2(\eta)} \int_0^{\eta} \lambda^{N-1} \bar{\phi}_0^2(\lambda) d\lambda. \tag{B.1}$$

We also recall that:

$$\bar{\phi}_0(\xi) \sim -\frac{(N(N-2))^{\frac{N}{2}}}{4N} |\xi|^{-(N-2)} (1 + O(|\xi|^{-2})), \qquad \xi \to +\infty.$$

In space dimension $N = 3$ we just use the asymptotics of $\bar{\phi}_0$ and after some easy calculations we arrive at:

$$H(\xi) = \frac{\sqrt{3}}{8} \xi + O(1) \qquad \xi \to +\infty, \qquad N = 3, \tag{B.2}$$

and this is enough for our purposes in Section 3.

In space dimension $N = 4$ we have that:

$$\bar{u}(\xi) = \frac{8}{8 + \xi^2}, \qquad \bar{\phi}_0(\xi) = -\frac{4(\xi^2 - 7)}{(\xi^2 + 8)^2} \sim -\frac{4}{\xi^2} \left(1 + O(\frac{1}{\xi^2})\right), \qquad \xi \to +\infty.$$

Plugging the exact value of $\bar{\phi}_0$ in the integral in (B.1) and changing variables to $\zeta = \lambda^2$, $\nu = \eta^2$, we write:

$$f(\xi) = -\frac{1}{4} \int_0^{\xi^2} \frac{(\nu + 8)^4 d\nu}{\nu^2 (\nu - 8)^2} \int_0^{\nu} \frac{\zeta (\zeta - 8)^2 d\zeta}{(\zeta + 8)^4}.$$

The second integral is then calculated explicitly:

$$L(\nu) = \int_0^{\zeta} \frac{\zeta (\zeta - 8)^2 d\zeta}{(\zeta + 8)^4} = \log\left(\frac{\nu + 8}{8}\right) - \frac{7\nu^3 + 48\nu^2 + 192\nu}{3(\nu + 8)^3}.$$

Hence:

$$L(\nu) = \log(\nu) - \left(\frac{7}{3} + \log 8\right) + O(\frac{1}{\nu}), \qquad \nu \to +\infty.$$

As to the integrand in the first integral, we write:

$$\frac{(\nu+8)^4}{\nu^2(\nu-8)^2} = 1 + O(\frac{1}{\nu}), \qquad \nu \to +\infty.$$

Consequently we have:

$$f(\xi) \sim -\frac{1}{4}\int_0^{\xi^2} \left(\log\nu - \left(\frac{7}{3} + \log 8\right) + O(\frac{\log\nu}{\nu})\right) d\nu, \qquad \xi \to +\infty,$$

whence:

$$f(\xi) = -\frac{1}{2}\xi^2\log\xi + \frac{1}{4}\left(\frac{10}{3} + \log 8\right)\xi^2 + O(\log^2\xi), \qquad \xi \to +\infty.$$

Using the asymptotics of $\bar{\phi}_0$ we finally obtain:

$$H(\xi) = 2\log\xi - \left(\frac{10}{3} + \log 8\right) + O(\frac{\log^2\xi}{\xi^2}), \qquad \xi \to +\infty, \qquad N = 4. \tag{B.3}$$

We next compute the asymptotics of $H(\xi)$ in the case $N \geq 5$. Using the asymptotics of $\bar{\phi}_0$ (cf Section 2), we write for the second integral in (B.1):

$$\begin{aligned}\int_0^\eta \lambda^{N-1}\bar{\phi}_0^2(\lambda)d\lambda &= \int_0^\infty \lambda^{N-1}\bar{\phi}_0^2(\lambda)d\lambda - \int_\eta^\infty \lambda^{N-1}O(\frac{1}{\lambda^{2N-4}})d\lambda \\ &= \int_0^\infty \lambda^{N-1}\bar{\phi}_0^2(\lambda)d\lambda + O(\frac{1}{\eta^{N-4}}), \qquad \eta \to +\infty.\end{aligned}$$

Taking advantage of the exact value of $\bar{\phi}_0$ we compute:

$$I_N = \int_0^\infty \lambda^{N-1}\bar{\phi}_0^2(\lambda)d\lambda = \int_0^\infty \frac{\left(\frac{N-2}{4} - \frac{\lambda^2}{4N}\right)^2}{\left(1 + \frac{\lambda^2}{N(N-2)}\right)^N}\lambda^{N-1}d\lambda.$$

Changing variables to $\lambda = (N(N-2))^{1/2}x$, we obtain after some calculations:

$$I_N = \frac{(N-2)^2}{16}(N(N-2))^{\frac{N}{2}}B_N,$$

with:

$$B_N = \int_0^\infty \frac{(1-x^2)^2}{(1+x^2)^N}x^{N-1}dx = \frac{1}{2}\int_0^\infty \frac{(1-t)^2}{(1+t)^N}t^{\frac{N-1}{2}}dt. \tag{B.4}$$

We now return to function $f(\xi)$ in (B.1). Using the asymptotics of $\bar{\phi}_0$ and our previous calculations we may write:

$$f(\xi) \sim -\frac{16N^2}{(N(N-2))^N}\int_0^\xi \eta^{N-3}d\eta\left(I_N + O(\frac{1}{\eta^{N-4}})\right), \qquad \xi \to +\infty.$$

Hence:

$$f(\xi) = -\frac{16N^2 I_N}{N^N(N-2)^{N+1}}\xi^{N-2} + O(\xi^2), \qquad \xi \to +\infty.$$

Using again the asymptotics of $\bar{\phi}_0$ as well as the value of I_N, we finally arrive at:

$$H(\xi) = \frac{N(N-2)B_N}{4} + O(\frac{1}{\xi}), \qquad \xi \to +\infty, \qquad N \geq 5, \tag{B.5}$$

with B_N as given in (B.4).

C. Some drawings of $u(x,t)$ near blowup

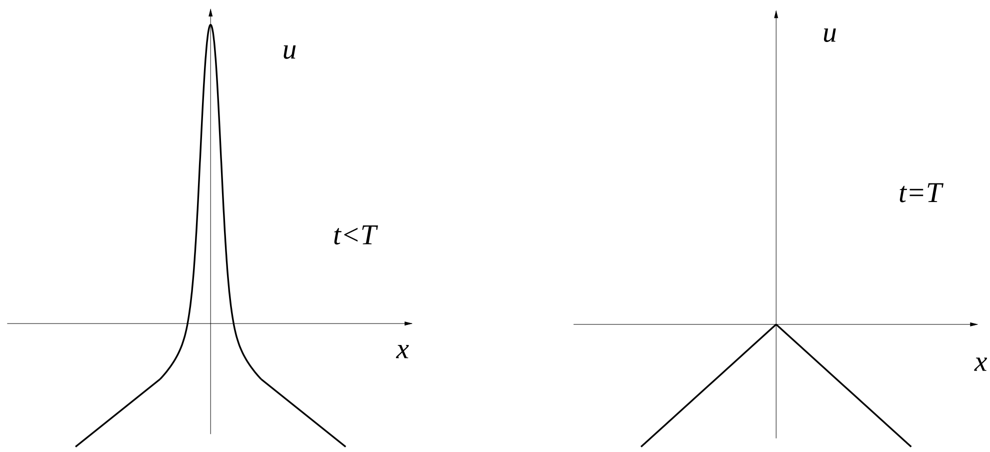

Figure 1: Case $N = 3$, $k = 1$.

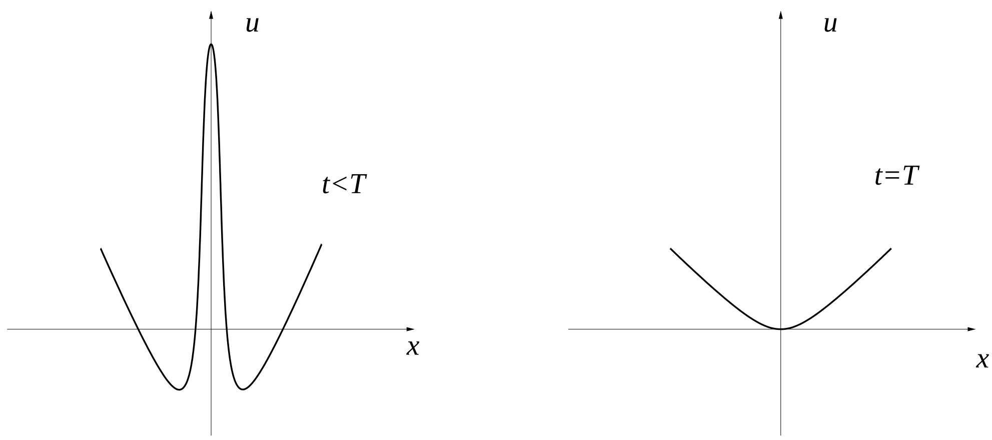

Figure 2: Case $N = 3$, $k = 2$.

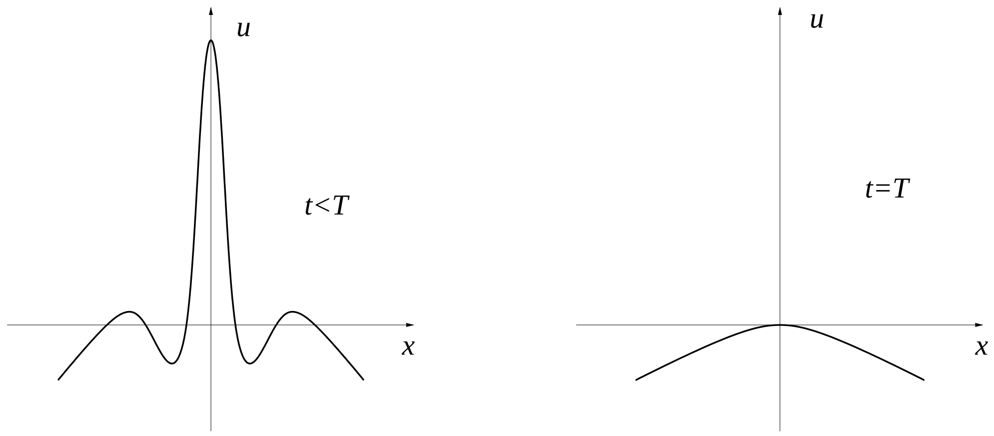

Figure 3: Case $N = 3$, $k = 3$.

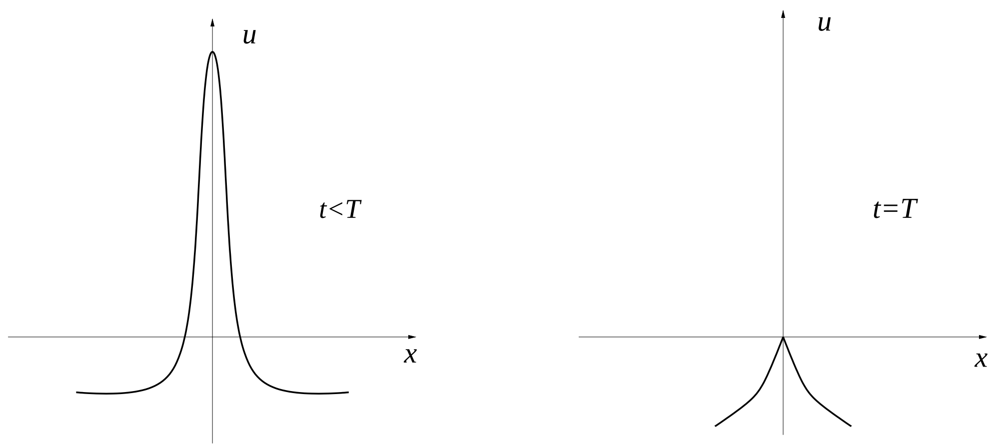

Figure 4: Case $N = 4$, $l = 0$.

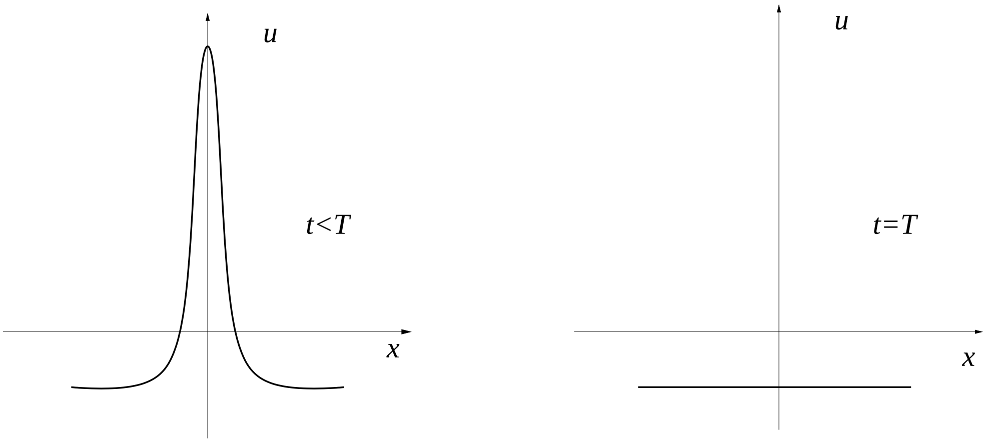

Figure 5: Case $N = 5$, $l = 0$.

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