Arch. Rational Mech. Anal. 229 (2018) 1281–1286
Digital Object Identifier (DOI) https://doi.org/10.1007/s00205-018-1264-y

CrossMark

Correction to: Sharp Trace Hardy–Sobolev–Maz'ya Inequalities and the Fractional Laplacian

Stathis Filippas, Luisa Moschini & Achilles Tertikas

Communicated by F. Lin

Correction to: Arch. Rational Mech. Anal. 208 (2013) 109–161
https://doi.org/10.1007/s00205-012-0594-4

We became aware of a gap in the proof of Theorems 2(iii) and 6(ii) of [1] in the case where $a = 1 - 2s \in (0, 1)$. We thank Arka Mallcik for bringing this to our attention. We used there an L^1 weighted trace Sobolev inequality, namely the displayed formula below relation (5.10) in page 143, which is valid for $a \in (-1, 0]$. We provide a proof for the case $a \in (0, 1)$ using instead the following weighted trace inequality:

Theorem 1. *Let $a \in (0, 1)$ and $1 + a < p < n + 1$. Then, there exists a positive constant c such that for all $u \in C_0^\infty(\mathbf{R}^n \times \mathbf{R})$ with $u(x, 0) = 0$, $x \in \mathbf{R}^n_-$,*

$$\int_0^{+\infty} \int_{\mathbf{R}^n_+} y^a |\nabla u|^p \mathrm{d}x\mathrm{d}y \geq c \left(\int_{\mathbf{R}^n_+} |u(x, 0)|^{\frac{pn}{n+1+a-p}} \mathrm{d}x \right)^{\frac{n+1+a-p}{n}} .$$

Proof. We start with the standard trace inequality

$$\int_{\mathbf{R}^n_+} |u(x, 0)| \mathrm{d}x \leq \int_0^{+\infty} \int_{\mathbf{R}^n_+} |\nabla u| \mathrm{d}x\mathrm{d}y.$$

For $q := \frac{pn}{n+1+a-p} > p$, we have

The original article can be found online at https://doi.org/10.1007/s00205-012-0594-4.

$$\begin{aligned}\int_{\mathbf{R}^n_+} |u(x,0)|^q \mathrm{d}x &\le q \int_0^{+\infty} \int_{\mathbf{R}^n_+} |u|^{q-1} |\nabla u| \mathrm{d}x\mathrm{d}y \\ &= q \int_0^{+\infty} \int_{\mathbf{R}^n_+} y^{\frac{a}{p}} |\nabla u| \;\; y^{-\frac{a}{p}} |u|^{q-1} \mathrm{d}x\mathrm{d}y \\ &\le q \left(\int_0^{+\infty} \int_{\mathbf{R}^n_+} y^a |\nabla u|^p \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{p}} \\ &\qquad \left(\int_0^{+\infty} \int_{\mathbf{R}^n_+} y^{-\frac{a}{p-1}} |u|^{\frac{(q-1)p}{p-1}} \mathrm{d}x\mathrm{d}y \right)^{\frac{p-1}{p}} .\end{aligned}$$

The result then follows using the Sobolev inequality of Corollary 2, page 139 of [2]. □

We will also use the following variant of Lemma 11 of [1]:

Lemma 1. *Let $A > 0$, $B + 1 > 0$ and $A + B + 2 > 2\Gamma > 0$. Then, there exists a positive constant c such that for all $v \in C_0^\infty(\mathbf{R}^n \times \mathbf{R})$ the following inequality holds true*:

$$c \int_0^{+\infty} \int_{\mathbf{R}^n_+} \frac{y^{A-1} x_n^B}{(x_n^2 + y^2)^{\Gamma - \frac{1}{2}}} |v| \mathrm{d}x\mathrm{d}y \le \int_0^{+\infty} \int_{\mathbf{R}^n_+} \frac{y^A x_n^{1+B}}{(x_n^2 + y^2)^{\Gamma}} |\nabla v| \mathrm{d}x\mathrm{d}y. \quad (1)$$

The same result holds true if we replace $\mathbf{R}^n_+$ by $\mathbf{R}^n_-$ with $|x_n|$ in place of x_n.

Proof. Using polar coordinates and the fact that, for $\theta \in (0, \frac{\pi}{2})$,

$$A(\sin\theta)^{A-1}(\cos\theta)^B = (1 + A + B)(\sin\theta)^{A+1}(\cos\theta)^B + \frac{\mathrm{d}}{\mathrm{d}\theta}((\sin\theta)^A(\cos\theta)^{1+B}),$$

we get

$$\begin{aligned} A \int_0^{\frac{\pi}{2}} (\sin\theta)^{A-1}(\cos\theta)^B |v| \mathrm{d}\theta &\le (1 + A + B) \int_0^{\frac{\pi}{2}} (\sin\theta)^{1+A}(\cos\theta)^B |v| \mathrm{d}\theta \\ &\quad + \int_0^{\frac{\pi}{2}} (\sin\theta)^A (\cos\theta)^{1+B} |v_\theta| \mathrm{d}\theta. \end{aligned} \quad (2)$$

We next multiply (2) by $r^{A+B+1-2\Gamma}$ and then integrate over $(0, \infty)$ to get

$$\begin{aligned} A &\int_0^{+\infty} \int_0^{+\infty} \frac{y^{A-1} x_n^B}{(x_n^2 + y^2)^{\Gamma - \frac{1}{2}}} |v| \mathrm{d}x_n \mathrm{d}y \\ &\le (1 + A + B) \int_0^{+\infty} \int_0^{+\infty} \frac{y^{1+A} x_n^B}{(x_n^2 + y^2)^{\Gamma + \frac{1}{2}}} |v| \mathrm{d}x_n \mathrm{d}y \\ &\quad + \int_0^{+\infty} \int_0^{+\infty} \frac{y^A x_n^{1+B}}{(x_n^2 + y^2)^{\Gamma}} |\nabla v| \mathrm{d}x_n \mathrm{d}y. \end{aligned}$$

We conclude as in the proof of Lemma 11 of [1]. □

Using the previous lemma with $|v|^p$ in place of $|v|$ we easily get

Lemma 2. *Let $A > 0$, $B + 1 > 0$, $A + B + 2 > 2\Gamma > 0$ and $p \geq 1$. Then, there exists a positive constant c such that for all $v \in C_0^\infty(\mathbf{R}^n \times \mathbf{R})$ the following inequality holds true*:

$$\int_0^{+\infty}\int_{\mathbf{R}^n_+} \frac{y^{A+p-1}x_n^{p+B}}{(x_n^2+y^2)^{\Gamma+\frac{p-1}{2}}}|\nabla v|^p \mathrm{d}x\mathrm{d}y \geq c\int_0^{+\infty}\int_{\mathbf{R}^n_+} \frac{y^{A-1}x_n^B}{(x_n^2+y^2)^{\Gamma-\frac{1}{2}}}|v|^p \mathrm{d}x\mathrm{d}y.$$

The same result holds true if we replace $\mathbf{R}^n_+$ by $\mathbf{R}^n_-$ with $|x_n|$ in place of x_n.

We are now ready to give the proof of Theorem 2 part (iii) in case $a \in (0, 1)$.

Proof of Theorem 2(iii). Our aim is to establish

$$\int_0^{+\infty}\int_{\mathbf{R}^n_+} y^a\phi^2|\nabla w|^2 \mathrm{d}x\mathrm{d}y \geq c\left(\int_{\mathbf{R}^n_+}|(\phi w)(x,0)|^{\frac{2n}{n+a-1}}\mathrm{d}x\right)^{\frac{n+a-1}{n}}, \tag{3}$$

where ϕ is given by Lemma 2 of [1]. We recall that $\phi(x, 0) = 1$, $x \in \mathbf{R}^n_+$.

For $a \in (0, 1)$ and p such that

$$1 + \frac{ap}{2} < p < 2 \quad \Leftrightarrow \quad \frac{2}{2-a} < p < 2,$$

Theorem 1 gives

$$\int_0^{+\infty}\int_{\mathbf{R}^n_+} y^{\frac{ap}{2}}|\nabla u|^p \mathrm{d}x\mathrm{d}y \geq c\left(\int_{\mathbf{R}^n_+}|u(x,0)|^Q \mathrm{d}x\right)^{\frac{p}{Q}}, \tag{4}$$

with

$$Q = \frac{2pn}{2(n+1) - p(2-a)} > p.$$

We apply (4) to $u = \phi^\theta v$, with

$$\theta = 1 + \frac{(2-p)(n+1)}{p(n+a-1)} > 1,$$

to obtain

$$\int_0^{+\infty}\int_{\mathbf{R}^n_+} y^{\frac{ap}{2}}\phi^{\theta p}|\nabla v|^p \mathrm{d}x\mathrm{d}y + \theta^p\int_0^{+\infty}\int_{\mathbf{R}^n_+} y^{\frac{ap}{2}}|\nabla\phi|^p\phi^{(\theta-1)p}|v|^p \mathrm{d}x\mathrm{d}y$$
$$\geq c\left(\int_{\mathbf{R}^n_+}|v(x,0)|^Q \mathrm{d}x\right)^{\frac{p}{Q}}. \tag{5}$$

We next show that in the above inequality the second term of the left-hand side is controlled by the first one. Using the asymptotics of ϕ from Lemma 2 of [1] this is equivalent to

$$\int_0^{+\infty}\int_{\mathbf{R}^n_+}\frac{y^{\frac{ap}{2}}x_n^{\theta p}}{(x_n^2+y^2)^{\frac{(2+a)\theta p}{4}}}|\nabla v|^p\mathrm{d}x\mathrm{d}y \geq c\int_0^{+\infty}\int_{\mathbf{R}^n_+}\frac{y^{-\frac{ap}{2}}x_n^{(\theta-1)p}}{(x_n^2+y^2)^{\sigma}}|v|^p\mathrm{d}x\mathrm{d}y, \tag{6}$$

with $\sigma=\frac{(2-a)p}{4}+\frac{(2+a)(\theta-1)p}{4}$. To prove this we apply Lemma 2 with $A=1-\frac{ap}{2}>0$, $B=(\theta-1)p$, and $\Gamma=\frac{1}{2}+\sigma=\frac{1}{2}+\frac{(2-a)p}{4}+\frac{(2+a)(\theta-1)p}{4}$, noting that

$$A+B+2-2\Gamma=\frac{(2-p)(2-a)(n-1)}{2(n+a-1)}>0.$$

We thus get

$$\int_0^{+\infty}\int_{\mathbf{R}^n_+}\frac{y^{\frac{ap}{2}+p(1-a)}x_n^{\theta p}}{(x_n^2+y^2)^{\frac{p}{2}(1-a)+\frac{2+a}{4}\theta p}}|\nabla v|^p\mathrm{d}x\mathrm{d}y \geq c\int_0^{+\infty}\int_{\mathbf{R}^n_+}\frac{y^{-\frac{ap}{2}}x_n^{(\theta-1)p}}{(x_n^2+y^2)^{\sigma}}|v|^p\mathrm{d}x\mathrm{d}y,$$

which implies (6), since $\frac{y}{(y^2+x_n^2)^{1/2}}<1$. From (5) and (6) we have

$$\int_0^{+\infty}\int_{\mathbf{R}^n_+}y^{\frac{ap}{2}}\phi^{\theta p}|\nabla v|^p\mathrm{d}x\mathrm{d}y\geq c\left(\int_{\mathbf{R}^n_+}|v(x,0)|^Q\mathrm{d}x\right)^{\frac{p}{Q}}. \tag{7}$$

We set $v=|w|^\theta$, we note that $\theta Q=\frac{2n}{n+a-1}$, and then apply Hölder's inequality to get

$$\begin{aligned} c\left(\int_{\mathbf{R}^n_+}|w(x,0)|^{\frac{2n}{n+a-1}}\mathrm{d}x\right)^{\frac{p}{Q}} &\leq \int_0^{+\infty}\int_{\mathbf{R}^n_+}y^{\frac{ap}{2}}\phi^p|\nabla w|^p\,(\phi|w|)^{p(\theta-1)}\mathrm{d}x\mathrm{d}y \\ &\leq \left(\int_0^{+\infty}\int_{\mathbf{R}^n_+}y^a\phi^2|\nabla w|^2\mathrm{d}x\mathrm{d}y\right)^{\frac{p}{2}} \\ &\qquad \left(\int_0^{+\infty}\int_{\mathbf{R}^n_+}(\phi|w|)^{\frac{2(n+1)}{n+a-1}}\mathrm{d}x\mathrm{d}y\right)^{\frac{2-p}{2}}. \end{aligned}$$

We conclude using the Sobolev inequality (5.10) of [1]. □

Similarly, we have

Proof of Theorem 6(ii). This time our aim is to establish

$$\int_0^{+\infty}\int_{\mathbf{R}^n} y^a\phi^2|\nabla w|^2 \mathrm{d}x\mathrm{d}y \geq c\left(\int_{\mathbf{R}^n_+} |w(x,0)|^{\frac{2n}{n+a-1}}\mathrm{d}x\right)^{\frac{n+a-1}{n}}, \tag{8}$$

where ϕ is now given by Lemma 4 of [1]. Working as in the previous proof with the same choices of θ, p and Q we arrive at the analogue of (5), which is

$$\int_0^{+\infty}\int_{\mathbf{R}^n} y^{\frac{ap}{2}}\phi^{\theta p}|\nabla v|^p \mathrm{d}x\mathrm{d}y + \theta^p\int_0^{+\infty}\int_{\mathbf{R}^n} y^{\frac{ap}{2}}|\nabla\phi|^p\phi^{(\theta-1)p}|v|^p\mathrm{d}x\mathrm{d}y$$
$$\geq c\left(\int_{\mathbf{R}^n_+}|v(x,0)|^Q\mathrm{d}x\right)^{\frac{p}{Q}}. \tag{9}$$

We again need to control the second term of the left-hand side by the first one. To establish this we split the integrals over $\mathbf{R}^n_+$ and $\mathbf{R}^n_-$. Using the asymptotics of ϕ from Lemma 4 of [1], the required inequality on $\mathbf{R}^n_+$ reads

$$\int_0^{+\infty}\int_{\mathbf{R}^n_+}\frac{y^{\frac{ap}{2}}}{(x_n^2+y^2)^{\frac{a\theta p}{4}}}|\nabla v|^p\mathrm{d}x\mathrm{d}y$$
$$\geq c\int_0^{+\infty}\int_{\mathbf{R}^n_+}\frac{y^{-\frac{ap}{2}}}{(x_n^2+y^2)^{\sigma}}|v|^p\mathrm{d}x\mathrm{d}y, \tag{10}$$

with $\sigma = \frac{(2-a)p}{4}+\frac{a(\theta-1)p}{4}$. To prove this we apply Lemma 2 with $A = 1-\frac{ap}{2} > 0$, $B = 0$, and $\Gamma = \frac{1}{2}+\sigma = \frac{1}{2}+\frac{(2-a)p}{4}+\frac{a(\theta-1)p}{4}$, noting that

$$A+B+2-2\Gamma = \frac{(2-p)(2-a)(n-1)}{2(n+a-1)} > 0.$$

We thus get

$$\int_0^{+\infty}\int_{\mathbf{R}^n_+}\frac{y^{\frac{ap}{2}+p(1-a)}x_n^p}{(x_n^2+y^2)^{\frac{p}{2}(1-a)+\frac{p}{2}+\frac{a\theta p}{4}}}|\nabla v|^p\mathrm{d}x\mathrm{d}y$$
$$\geq c\int_0^{+\infty}\int_{\mathbf{R}^n_+}\frac{y^{-\frac{ap}{2}}}{(x_n^2+y^2)^{\sigma}}|v|^p\mathrm{d}x\mathrm{d}y,$$

which implies (10).

The required inequality over $\mathbf{R}^n_-$ is equivalent to

$$\int_0^{+\infty}\int_{\mathbf{R}^n_-}\frac{y^{\frac{ap}{2}+(1-a)\theta p}}{(x_n^2+y^2)^{\frac{(2-a)\theta p}{4}}}|\nabla v|^p\mathrm{d}x\mathrm{d}y$$
$$\geq c\int_0^{+\infty}\int_{\mathbf{R}^n_-}\frac{y^{-\frac{ap}{2}+(1-a)(\theta-1)p}}{(x_n^2+y^2)^{\frac{(2-a)\theta p}{4}}}|v|^p\mathrm{d}x\mathrm{d}y.$$

This is proved once again by applying Lemma 2 with $A = 1-\frac{ap}{2}+(1-a)(\theta-1)p > 0$, $B = 0$, and $\Gamma = \frac{1}{2} + \frac{(2-a)\theta p}{4}$, noting that

$$A + B + 2 - 2\Gamma = \frac{(2-p)(2-a)(n-1)}{2(n+a-1)} > 0.$$

To conclude we argue as in the previous case of the Proof of Theorem 2(iii). We omit further details. □

References

1. FILIPPAS, S., MOSCHINI, L., TERTIKAS, A.: Sharp trace Hardy–Sobolev–Maz'ya inequalities and the fractional Laplacian. *Arch. Ration. Mech. Anal.* **208**, 109–161 (2013)
2. MAZ'YA, V.: *Sobolev spaces with pplications to elliptic partial differential equations, second, revised and augmented edition*. Springer, Heidelberg, 2011

STATHIS FILIPPAS
Department of Mathematics and Applied Mathematics,
University of Crete,
71409 Heraklion, Greece.
e-mail: filippas@uoc.gr

and

LUISA MOSCHINI
Dipartimento di Scienze di Base ed Applicate per l'Ingegneria,
University of Rome "La Sapienza",
00185 Rome, Italy.
e-mail: luisa.moschini@sbai.uniroma1.it

and

ACHILLES TERTIKAS
Department of Mathematics and Applied Mathematics,
University of Crete,
71409 Heraklion, Greece.
e-mail: tertikas@uoc.gr

(*Received December 6, 2017 / Accepted May 19, 2018*)
Published online June 8, 2018